Week 1
Orientation

In this section, you will be introduced to the online environment as well as course policies and expectations.

Objectives

- Students will be able to navigate the online system and record their answers using the online tools provided.
- Students will apply the knowledge learned in the "how to answer questions" tutorial, to effectively use the math palette tool to record answers.
- Students will be able to identify important concepts from the course syllabus.

MyMathLab for MyLabsPlus—Your Interactive Learning Environment

MyMathLab for MyLabsPlus engages students in active learning—it's modular, self-paced, accessible anywhere with Web access, and adaptable to each student's learning style. MyMathLab provides free-response exercises correlated directly to the textbook that regenerate algorithmically for unlimited practice and mastery, and in homework and practice modes, each exercise is accompanied by an interactive guided solution and sample problem. MyMathLab provides students with additional multimedia resources, such as video lectures, animations, and an eBook, to independently improve their understanding and performance. MyLabsPlus is the course management system that is used to access your MyMathLab course materials.

Online Log-In Directions for MyLabsPlus

Your username for MyLabsPlus is your Network Identification Number (NID). Find your NID at www.my.ucf.edu by clicking "What are my PID and NID?" Your initial password should have been mailed to your Knights email account. If the enrollment email is not in your Inbox, there are typically four explanations:

- You have deleted the email
- It is in your Junk or Spam folder
- You added the course during the add/drop week
- You do not have your Knights email account on file with MyUCF

In the first case, you can use the password retrieval system on the website. Go to www.ucf.mylabsplus.com and click on Forgot your password. Enter your NID as your user id. Your password will be sent to your knights email account. In the latter two cases, you should make sure your myUCF profile is current and send an email to your instructor explaining the situation.

To change your password in MyLabsPlus

- It is highly recommended that you change your password to something more easily remembered the first time you log in.
- Upon submitting your username and password, you will be taken to a screen with a "My Profile" link in the top right corner.
- Click this link and you will be prompted to enter your current password and new password (twice).
- Please keep in mind that you'll need to have your password memorized when you attend each testing session throughout the semester.

Technical Support

While computers are provided in the Mathematics Assistance and Learning Lab (MALL) for students to work on assignments, we understand many students desire to work on their personal computers as well. Should this be the case for you, please understand your instructor is not, and cannot be your personal technical support line. Should you encounter problems accessing anything in MyLabsPlus, please feel free to contact Pearson Technical Support at 1-888-883-1299. Although the support line is open 24 hours a day, seven days a week, the best time to call is Monday through Friday between 8:00am and 8:30pm.

Access Codes

While you will be able to access the MyLabsPlus portion of our course website (including the syllabus, discussion board, and faculty information), the MyMathLab portion of the website (which contains all homework, quizzes, and tests) will be inaccessible until you enter an access code.

An access code can be acquired in a variety of ways:

- Included with the purchase of a new 3 module textbook package available at the on-campus bookstore as well as many local off-campus bookstores
- Purchased online directly from Pearson while logging into the system

Purchasing the textbook and access code online via other websites is discouraged since many students accidentally purchase the incorrect items and then have difficulty obtaining a refund.

Temporary Access Codes

-
Please note that in an effort to get students started on their homework and quizzes as early as possible, a temporary access code is also available. This code is free, but it only lasts the first 21 days of the course.

After you log into MyLabPlus, there is a link in the navigation menu called Temporary Access. After clicking on the link, follow the directions to receive your complimentary 21 days of access.

To work on a homework assignment

1. Go to the Homework and Tests page.
2. Check the Due column for the assignment you want to work on.
   - If a flag icon appears to the left of the due date, then you must complete a prerequisite assignment before you can begin work on this one. Position your mouse pointer over the flag to display information about the prerequisite.
3. If the assignment has no prerequisites, then click the assignment link.
   - The Homework Overview page appears and gives you information about the assignment.
4. Click a question link to begin.
   - If you are redoing an assignment to improve your score, you see the correct answer to the current version question. Click Similar Exercise to generate another version of the question with different values or Try Again to refresh the same version of the question. You can answer the new version of the question to get more practice or try to improve your score.
5. Enter an answer and then click Check Answer.
   - If you answered correctly, a congratulatory message appears. If you answered incorrectly, a message will prompt you to try again.
   - If the question has multiple parts, the Check Answer button may change to read Continue. If so, click Continue to keep stepping through the problem, checking your answer after each step.
6. Complete the question. Your score on the assignment is automatically updated each time you complete a question.
7. Use the navigation controls in the player to move to a new question and continue working on the assignment.

Textbook Section 1.1

Linear Equations

Objectives

- Students will be able to solve a linear equation
- Students will be able to solve for a specified variable
- Students will be able to use the simple interest formula to calculate interest

Key Concepts

Linear equation

Written in the form $ax + b = 0$, where $a, b$ are real numbers, $a \neq 0$, $x$ is to the first power

Three Types of Equations

1. Conditional results in a single solution
2. Identity results in an infinite number of solutions or all real numbers
3. Contradiction - there is no solution

Textbook Section 1.2

Applications and Modeling with Linear Equations

Objectives

- Student will be able to solve an applied problem involving unknown numbers and geometry
- Student will be able to solve an applied problem involving motion
- Student will be able to solve an applied problem involving mixture
- Student will be able to solve an applied problem involving interest
- Student will be able to solve modeling problems

Key Concepts
Formulas

- Motion problems: rate \times time = distance
- Mixture problems: strength(%) \times quantity = amount pure
- Interest problems: interest rate \times principle = interest

Steps for Solving an Applied Problem

1. Read the problem.
2. Assign a variable.
3. Write an equation.
4. Solve.
5. State the answer.
6. Check.

Assignments

Syllabus, Schedule, and Protocols Quiz

By now, you should have reviewed the syllabus, schedule, and protocols under the "Start Here" link. Take the MAC1105 Syllabus, Schedule, and Protocols Quiz. You will find the quiz in the Table of Contents for this week or by selecting the Assessment link in the Course Tools menu. You have an unlimited number of attempts to take the quiz.

How to Enter Answers Tutorial

You will be using a math palette to enter your answers for your MyMathLab assignments. The tutorial provides information on the Player window, entering answers, math palette, graphing tool, doing homework, taking tests, and getting more help. It will take approximately 12 minutes to complete the tour. Go to http://media.pearsoncmg.com/college/pmlabs/player_tour/enteranswers.html to complete the tour.

Online Homework and Quiz Assignments

After completing the Syllabus, Schedule, and Protocols Quiz and the How to Enter Answers Tutorial, you are ready to begin working in the Interactive Learning Environment. Remember to refer back to the directions found in this week's information and to call technical support with any technical questions.

Click on the "MyLabsPlus" link below. Enter your NID (Network ID from UCF) and MyLabsPlus Password. If you cannot remember your MyLabsPlus Password, click on the Forgot your Password/User ID link on the site. To log into MyLabsPlus go to www.ucf.mylabsplus.com and begin working on your homework and quiz for this week.

Reminders

It is very important that you change your password for the MyLabsPlus system. The initial password is a case sensitive “strong” password that is often difficult to remember. When using a computer in the MALL, your password will not be saved and you will not be able to access your course materials, including your test.
Linear Equations, 1.1

Definition: A linear equation in one variable is an equation that can be written in the form:

\[ ax + b = 0 \]

where \( a \) and \( b \) are real numbers with \( a \neq 0 \).

- \( x \) is to the first power (\( x^1 \))
- first degree equation

Addition & Multiplication Properties of Equality

For real numbers \( a, b, \) and \( c, \)

- If \( a = b \), then \( a + c = b + c \)

Example:

If \( x = 4 \),
then \( x + 3 = 4 + 3 \)

The same number may be added to both sides of an equation without changing the solution set.

- If \( ac = bc \) and \( c \neq 0 \), then \( a = b \)

Example:

If \( 2x = 8 \),
then \( x = 4 \)

We can divide both sides of an equation by any non-zero number and the solution set will stay the same.

Example #12

Solve the equation for \( x \):

\[ 4(-2x + 1) = 6 - (2x - 4) \]

Solution:

We distribute on both sides of the equation:

\[-8x + 4 = 6 - 2x + 4 \]

and collect terms:

\[-8x + 4 = 10 - 2x \]

We then add \( 2x \) to both sides:

\[-8x + 4 + 2x = 10 - 2x + 2x \]

\[-6x + 4 = 10 \]

We subtract \( 4 \) from both sides:

\[-6x + 4 - 4 = 10 - 4 \]

\[-6x = 6 \]

and divide both sides by \(-6:\)

\[ -6x = 6 \]
\[ \frac{-6x}{-6} = \frac{6}{-6} \]
\[ x = -1 \]

so this is our solution.

Example: #14

Solve for \( x \):

\[ \frac{7}{4} + \frac{1}{2}x - \left(\frac{3}{2}\right) = \frac{4}{5}x \]

Solution:
We could start adding and subtracting, but it will be easier if we clear fractions first by multiplying by the LCD of all the denominators here, which is 20. We multiply both sides by 20.

\[
\begin{align*}
20\left(\frac{7}{4} + \frac{1}{5} - \frac{3}{2}\right) &= 20\left(\frac{4}{5}\right) \\
35 + 4x - 30 &= 16x \\
5 + 4x &= 16x \\
5 &= 12x \\
\end{align*}
\]

We gather terms:

\[
\begin{align*}
5 + 4x &= 16x \\
5 &= 12x \\
\end{align*}
\]

and divide by 12 to solve for \(x\):

\[
x = \frac{5}{12}
\]

There are three types of linear equations you will come across: The first is an identity: attempting to solve the equation leads to a statement which is always true, such as \(0 = 0\). The solution set is all real numbers, sometimes written as \([\text{all real numbers}]\), since the statement is true no matter which \(x\) you choose.

**Example #34**

Solve for \(x\):

\[-0.6(x - 5) + 0.6(x - 6) = 0.2x - 1.8\]

**Solution:**

Like clearing fractions, it may be useful to clear decimals by multiplying by 10 first:

\[-6(x - 5) + 6(x - 6) = 2x - 18\]

We then distribute:

\[-6x + 30 + 6x - 36 = 2x - 18\]

and collect terms:

\[2x - 18 = 2x - 18\]

Now, we add 18 to both sides:

\[2x - 18 + 18 = 2x - 18 + 18\]

\[2x = 2x\]

Finally, we subtract 2x from both sides:

\[2x - 2x = 2x - 2x\]

\[0 = 0\]

This is always true. Therefore, the original equation was an identity and the solution set is \([\text{all real numbers}]\).

**Example #32**

Solve for \(x\):

\[-8(x + 5) = -8x - 5(x + 1)\]

**Solution:**

We solve as usual:

\[
\begin{align*}
-8(x + 5) &= -8x - 5(x + 1) \\
-8x - 24 &= -8x - 5x - 5 \\
-8x - 24 &= -13x - 5 \\
5x &= -5 \\
5x &= 5 \\
x &= \frac{19}{5}
\end{align*}
\]
In other words, the original equation is only true for this one value. We may write this as:

\[ x = \left[ \frac{19}{5} \right] \]

A contradiction: attempting to solve the equation leads to a false statement, such as 3 = 7. The solution is the empty set, denoted \( \emptyset \).

**Example #36**

Solve for \( x \):

\[-6(2x + 1) - 3(x - 4) = -15x + 1\]

Solution:

We go through the usual steps:

\[-6(2x + 1) - 3(x - 4) = -15x + 1\]

\[-12x - 6 - 3x + 12 = -15x + 1\]

\[-15x + 6 = -15x + 1\]

\[6 = 1\]

This is always false, no matter which \( x \) we choose! Therefore, the answer is the empty set, \( \emptyset \).

**Example #42**

Solve for \( w \) in:

\[ P = 2l + 2w \]

Solution:

We subtract 2\( l \) from both sides of the equation:

\[ P - 2l = 2l + 2w - 2l \]

\[ P - 2l = 2w \]

and divide both sides of the equation by 2:

\[ \frac{P - 2l}{2} = \frac{2w}{2} \]

\[ w = \frac{P - 2l}{2} \]

**Example #47**

Solve for \( h \) in:

\[ S = 2lw + 2wh + 2hl \]

Solution:

We begin by moving all terms which do not contain \( h \) to the left-hand side of the equation:

\[ S - 2lw = 2lw + 2wh + 2hl - 2lw \]

\[ S - 2lw = 2wh + 2hl \]

Now, since both of the terms on the right-hand side contain \( h \), we can factor out the coefficients as follows:

\[ S - 2lw = (2w + 2h)h \]

Now, we divide by the coefficient of \( h \):

\[ \frac{S - 2lw}{2w + 2l} = h \]

We have not solved for a particular number, but we have solved for \( h \).
Simple Interest

\[ I = Prf \]
\[ I = \text{Simple Interest} \]
\[ P = \text{Principal} \]
\[ r = \text{annual interest rate (decimal)} \]
\[ t = \text{time (years)} \]

Problem #60

Jennifer Johnston borrows $20,000 from her bank to open a florist shop. She agrees to repay the money in 18 months with simple annual interest of 10.4%.

1. How much must she pay the bank in 18 months?

Jennifer has agreed to pay the bank back in 18 months, which is 1.5 years, so \( t = 1.5 \). Here, \( r = .104 \), not 10.4, since we need to put it in decimal form. The principal is the amount she borrowed, or 20,000.

Therefore, in 1.5 years, she will need to pay the original amount plus interest, or

\[ P + I = P + Prt \]
\[ = 20000 + 20000(.104)(1.5) \]
\[ = 24160.4 \]

So Jennifer must repay $24,160.40 at the end of the term.

1. How much of the amount in part a) is interest?

The interest is given by \( I = Prt = 20000(104)(1.5) = 31200.40 \).
Applications and Modeling with Linear Equations, 1.2

To solve an applied problem:
- **Step 1**: Read the problem.
- **Step 2**: Assign a variable.
- **Step 3**: Write an equation.
- **Step 4**: Solve the equation.
- **Step 5**: State the answer.
- **Step 6**: Check the solution.

**Type 1: Geometry**

#10: Blake Moore must build a rectangular storage shed. He wants the length to be 8 ft greater than the width, and the perimeter will be 44 ft. Find the length and the width of the shed.

Solution: After the first step, reading the problem, which we will take for granted from now on, we assign variables:
- \( L \) = length of storage shed
- \( W \) = width of storage shed
- \( P \) = perimeter of storage shed

Now, we know that for rectangles,
\[
P = 2L + 2W
\]
The problem has also told us that the perimeter of the shed should be 44 ft, so \( P = 44 \). We also know that:
\[
L = W + 8
\]
In other words, the length is 6 ft greater than the width. Therefore, we plug in everything we know:
\[
44 = 2(W + 6) + 2W
\]
This is just a linear equation, so we use the techniques of the last section to solve:
\[
44 = 2W + 12 + 2W
\]
\[
44 = 4W + 12
\]
\[
32 = 4W
\]
\[
W = 8
\]
So the width is 8 feet. We know that the length is 8 more than this, so the length is 14 feet. We can check – if a rectangle is 8×14, it has perimeter
\[
2(8) + 2(14) = 44\text{ ft}
\]
as required. Therefore, we say that the storage shed has dimensions of 8 ft by 14 ft.

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**Type 2: Distance**

\[d = rt\]
\[d = \text{distance}\]
\[r = \text{rate}\]
\[t = \text{time}\]

#28: Joe traveled against the wind in a small plane for 3 hours. The return trip with the wind took 2.8 hours. Find the speed of the wind if the speed of the plane in still air is 180 mph.

Solution: First, we assign our variables. At this point, we really only need:
- \( d \) = distance of both trips = unknown
- \( w \) = speed of the wind

For the trip against the wind, the rate of the plane is \( 180 - w \), since the plane is traveling against the wind. The time is 3 hours, as stated by the problem. Therefore, the distance traveled against the wind is
\[
d = rt = (180 - w)(3)
\]
Now, for the trip with the wind, the rate of the plane is \( 180 + w \), since the plane is being pushed along by the wind. The time is 2.8 hours, as stated by the problem. Therefore, the distance traveled with the wind is
\[
d = rt = (180 + w)(2.8)
\]
The key is to recognize that both of the distances are the same. We do not know yet what \( d \) is, but we know that Joe traveled the same distance with and against the wind. Therefore,
\[
(180 - w)(3) = (180 + w)(2.8)
\]
We distribute:
\[
540 - 3w = 504 + 2.8w
\]
And solve as usual
\[
36 - 3w = 2.8w
\]
36 = 5.8w

w = \frac{36}{5.8} \approx 6.21

Therefore, the speed of the wind is approximately 0.21 miles per hour.

Type 3: Mixture Problems

Strength (%) \times Amount of Solution = Amount of Pure

#30) Marin Caswell needs 10% hydrochloric acid for a chemistry experiment. How much 5% acid should she mix with 60 mL of 20% acid to get a 10% solution?

Solution: We begin with a table and fill in what we know:

<table>
<thead>
<tr>
<th>Type of Solution</th>
<th>Strength</th>
<th>Amount</th>
<th>Amount Pure</th>
</tr>
</thead>
<tbody>
<tr>
<td>10% hydrochloric acid</td>
<td>0.10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5% hydrochloric acid</td>
<td>0.05</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20% hydrochloric acid</td>
<td>0.20</td>
<td>60</td>
<td>0.20(60)</td>
</tr>
</tbody>
</table>

From the problem, we see that we want to know "how much 5% solution", so we call the amount x and multiply across:

<table>
<thead>
<tr>
<th>Type of Solution</th>
<th>Strength</th>
<th>Amount</th>
<th>Amount Pure</th>
</tr>
</thead>
<tbody>
<tr>
<td>10% hydrochloric acid</td>
<td>0.10</td>
<td>60 + x</td>
<td>0.10(60 + x)</td>
</tr>
<tr>
<td>5% hydrochloric acid</td>
<td>0.05</td>
<td>x</td>
<td>0.05(x)</td>
</tr>
<tr>
<td>20% hydrochloric acid</td>
<td>0.20</td>
<td>60</td>
<td>0.20(60)</td>
</tr>
</tbody>
</table>

Now, if we start with 60 mL of 20% solution and add x mL of 5% solution, we end up with 60 + x mL of 10% solution:

<table>
<thead>
<tr>
<th>Type of Solution</th>
<th>Strength</th>
<th>Amount</th>
<th>Amount Pure</th>
</tr>
</thead>
<tbody>
<tr>
<td>10% hydrochloric acid</td>
<td>0.10</td>
<td>60 + x</td>
<td>0.10(60 + x)</td>
</tr>
<tr>
<td>5% hydrochloric acid</td>
<td>0.05</td>
<td>x</td>
<td>0.05(x)</td>
</tr>
<tr>
<td>20% hydrochloric acid</td>
<td>0.20</td>
<td>60</td>
<td>0.20(60)</td>
</tr>
</tbody>
</table>

Since we are adding the 5% solution and 20% solution together to get the 10%, solution, this gives us the equation

0.10(60 + x) = 0.05(x) + 0.20(60)

We can make solving the equation easier by clearing the decimal by multiplying both sides of the equation by 100:

10(60 + x) = 5(x) + 20(60)

Be very careful! A common miscalculation here is 100(0.05) = 50. This is a linear equation, so we solve with the techniques of the last section:

600 + 10x = 5x + 1200

5x = 600

x = 120

Therefore, we add 120 mL of the 5% solution to obtain 180 mL of the 10% solution.

Type 4: Interest

#38) A church building fund has invested some money in two ways: part of the money at 4% interest and four times as much at 3.5%. Find the amount invested at each rate if the total annual income from interest is $3000.

Solution: When dealing with simple interest, we are working with the equation $I = Prt$. Since there are two investments here, we create a table:

<table>
<thead>
<tr>
<th>Type of investment</th>
<th>P</th>
<th>r</th>
<th>t</th>
</tr>
</thead>
<tbody>
<tr>
<td>4% interest</td>
<td>0.04</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>3.5% interest</td>
<td>0.035</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

We have chosen $I = 1$ since we know the annual income from interest, i.e., how much interest the fund receives in 1 year. Now, we do not know how much has been invested at 4%, but we know that four times that amount is invested at 3.5%. Therefore, we can call the amount invested at 4% $x$ and update the table:

<table>
<thead>
<tr>
<th>Type of investment</th>
<th>P</th>
<th>r</th>
<th>t</th>
</tr>
</thead>
<tbody>
<tr>
<td>4% interest</td>
<td>$x$</td>
<td>0.04</td>
<td>1</td>
</tr>
<tr>
<td>3.5% interest</td>
<td>4$x$</td>
<td>0.035</td>
<td>1</td>
</tr>
</tbody>
</table>
Now we multiply across:

<table>
<thead>
<tr>
<th>Type of Investment</th>
<th>( l )</th>
<th>( P )</th>
<th>( r )</th>
<th>( t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4% interest</td>
<td>( x(0.04)(1) )</td>
<td>( x )</td>
<td>0.04</td>
<td>1</td>
</tr>
<tr>
<td>3.5% interest</td>
<td>( 4x(0.035)(1) )</td>
<td>( 4x )</td>
<td>0.035</td>
<td>1</td>
</tr>
</tbody>
</table>

Since the total amount of interest income is $3,600, we must have

\[3600 = x(0.04)(1) + 4x(0.035)(1)\]

This is a linear equation, so we can solve as usual:

\[3600 = 0.04x + 0.14x\]
\[3600 = 0.18x\]
\[x = \frac{3600}{0.18} = 20,000\]

Therefore, the church invests $20,000 at 4% interest and four times that amount, or $80,000 at 3.5% interest.

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**Modeling:**

**#41 Indoor Air Quality and Control**

The excess lifetime cancer risk \( R \) is a measure of the likelihood that an individual will develop cancer from a particular pollutant. For example, if \( R = 0.01 \) then a person has a 1% increased chance of developing cancer during a lifetime. (This would translate into 1 case of cancer for every 100 people during an average lifetime.) The value of \( R \) for formaldehyde, a highly toxic indoor air pollutant, can be calculated using the linear model \( R = kd \), where \( k \) is a constant and \( d \) is the daily dose in parts per million. The constant \( k \) for formaldehyde can be calculated using the formula

\[ k = \frac{0.152B}{W} \]

where \( B \) is the total number of cubic meters of air a person breathes in one day and \( W \) is a person's weight in kilograms.

a. Find \( k \) for a person who breathes in 20 m\(^3\) of air per day and weighs 75 kg.

We calculate

\[ k = \frac{0.152(20)}{75} = 0.0352 \]

b. Mobile homes in Minnesota were found to have a mean daily dose \( d \) of 0.42 part per million. Calculate \( R \) using the values of \( k \) found in part a).

We calculate

\[ R = kd = (0.0352)(0.42) = 0.014784 \]

c. For every 5000 people, how many cases of cancer could be expected each year from these levels of formaldehyde? Assume an average life expectancy of 72 yrs.

We can use the value of \( R \) found in part b) and recall that

\[ R = 0.014784 \] would translate to a 1.4784% increased chance of developing cancer during one's lifetime, and using 72 years:

\[ \text{number of cases} = \frac{5000(0.014784)}{72} = 1.02567 \]

Therefore, we can expect 1 case of cancer each year from these levels of formaldehyde.
Textbook Section 1.3
Complex Numbers

Objectives:
- The student will be able to write radical expression in \( a + bi \) form
- The student will be able to perform operations with complex numbers
- The student will be able to simplify powers of \( i \)
- The student will be able to find the complex conjugate

Key Concepts

Complex Numbers
- Set including real numbers & imaginary numbers
- Numbers of the form \( a + bi \) where \( a, b \) are real numbers.
- Imaginary unit is \( i = \sqrt{-1} \)
- Note: \( i^2 = -1 \)
- \( a = \) real part
  \( b = \) imaginary part

Note: Simplify using \( i \) before using other rules for radicals.

Operations with Complex Numbers

Adding and Subtracting Combine like terms
Multiplying No need to memorize the formula!
Multiply two binomials (FOIL) First Outside Inside Last
Dividing Multiply numerator and denominator by the conjugate of the denominator
Complex Conjugate of \( a + bi \) is \( a - bi \)

Textbook Section 1.4
Quadratic Equations

Objectives:
- The student will be able to solve a quadratic equation
- The student will be able to solve for an indicated variable
- The student will be able to use the discriminant to determine the number and type of solutions

Key Concepts

Quadratic Equation
- Standard form: \( ax^2 + bx + c = 0 \)
  \( a, b, c \) real numbers \( (a \neq 0) \)
  - The highest degree term is \( 2 (x^2) \)

Ways to solve:
1. Factoring
2. Square Root Method
3. Completing the Square
4. Quadratic Formula
   \[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

**Completing the Square Method**

\[ ax^2 + bx + c = 0 \quad [a \neq 0] \]

1. If \( a \neq 1 \), divide both sides of the equation by \( a \).
2. Move the constant term to the right-hand side.
3. Find \( \left( \frac{1}{2} b \right) \). Add this to both sides.
4. Factor the left-hand side.
5. Use the square root method to find the solution.

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**Textbook Section 1.5**

**Application and Modeling with Quadratic Equations**

**Objectives**

- The student will be able to solve problems involving unknown numbers
- The student will be able to solve problems involving perimeter, area, and volume
- The student will be able to solve applications involving the Pythagorean Theorem
- The student will be able to solve problems involving quadratic modeling

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**Handouts**

Some of the files you are about to view/download are PDF files. If you do not have Adobe Acrobat installed on your system, you can download the free Adobe Acrobat Reader at [http://www.adobe.com/products/acrobat/](http://www.adobe.com/products/acrobat/)

- Powers of i Handout
- Completing the Square Handout

---

**Assignments**

**Online Homework and Quiz Assignments**

After reviewing the Key Concepts and Handouts, log into MyLabsPlus and begin your homework and quiz for this week, go to [www.ufc.mylabsplus.com](http://www.ufc.mylabsplus.com) and begin working on your assignments.
How can we solve \( x^2 = -1 \)?

Definition: The imaginary unit is \( i = \sqrt{-1} \)

Definition: Complex numbers are numbers of the form \( a + bi \)

- \( a \) = real part
- \( b \) = imaginary part
- Set of numbers including Real numbers & Imaginary numbers

'Is 4 a complex number?'

Yes, 4 is a complex number; we can say that \( a = 4 \) and \( b = 0 \).

- \( a + bi = c + di \) if and only if \( a = b \) and \( c = d \)

'Two complex numbers are equal if and only if the real parts are equal and the imaginary parts are equal.'

The Expression \( \sqrt{-a} \):

Suppose \( a > 0 \) (a positive real number), then \( \sqrt{-a} = i\sqrt{a} \)

### #20) Simplify \( \sqrt{-15} \)

We write:

\[
\sqrt{-15} = i\sqrt{15}
\]

and note that no more simplification is possible.

### #24) Simplify \( -\sqrt{-80} \)

We write:

\[
-\sqrt{-80} = -(i\sqrt{80}) = -i\sqrt{80}
\]

As far as the negative sign under the radical is concerned, we have simplified sufficiently. However, we can go further:

\[
-i\sqrt{80} = -i\sqrt{16 \cdot 5} = -i(4\sqrt{5}) = -4i\sqrt{5}
\]

We can write this in various ways: \(-4i\sqrt{5}, -i\sqrt{5}, -4\sqrt{5}i\)

Sometimes we will avoid this last notation, as it may appear that the \( i \) is under the square root.

### #28) Simplify \( \sqrt{-5\sqrt{-15}} \)

It is vital to simplify in terms of \( i \) before using any of the other rules for radicals. We calculate:

\[
\sqrt{-5\sqrt{-15}} = i\sqrt{5} \cdot \sqrt{15}
\]

\[
i\sqrt{\sqrt{75}}
\]

\[
= i\sqrt{5\sqrt{3}}
\]

\[
= i(5\sqrt{3})
\]

\[
= -5\sqrt{3}
\]

### #38) Write in standard form:

\[
\frac{-9 - \sqrt{-18}}{3}
\]

Again, we use the rules for \( i \) before using any of the other simplification rules. We calculate:

\[
\frac{-9 - \sqrt{-18}}{3} = \frac{-9 - i\sqrt{18}}{3}
\]
We split this into two fractions:
\[-9 - \sqrt{12} \over 3\]
and simplify:
\[-3 - {\sqrt{2}} \over 3\]

Adding and Subtracting Complex Numbers:
For complex numbers \(a + bi\) and \(c + di\)
\[(a + bd) + (c + ad) = (a + c) + (b + d)i\]
\[(a + bd) - (c + ad) = (a - c) + (b - d)i\]
Add/subtract the real parts and add/subtract the imaginary parts.

#44) Calculate \((4 - i) + (8 + 5i)\)
Using the rule above, this is equal to:
\[(4 - i) + (8 + 5i) = (4 + 8) + (-1 + 5)i\]
\[= 12 + 4i\]

#50) Calculate \(3 - (4 - i) - 4i + (-2 + 5i)\)
We first distribute where necessary:
\[3 - (4 - 4i + (-2 + 5i)) = 3 - 4 + 4i - 2 + 5i\]
\[= (3-4-2) + (4 + 5)i\]
\[= -3 + 9i\]

Multiplication of Complex Numbers
For complex numbers \(a + bi\) and \(c + di\)
\[(a + bd) \cdot (c + ad) = (ac - bd) + (ad + bc)i\]
Note: Don't memorize this!
Multiply two binomials (F.O.I.L.) First Outside Inside Last.

#52) Calculate \((-2 + 3i)(4-2i)\)
We treat this as if it were a binomial and FOIL out:
\[(-2 + 3i)(4-2i) = (-2)(4) + (-2)(-2i) + (3i)(4) + (3i)(-2i)\]
\[= -8 + 4i + 12i - 6i^2\]
Now, we remember that \(i^2 = -1\)
\[= -8 + 4i + 12i - 6(-1)\]
\[= -8 + 4i + 12i + 6\]
#66) Calculate \(-5(4-3)^2\)

We must begin with the squared term:
\[-5(4-3)(4-3)\]
\[-5(16-12i-12i+9)\]
\[-5(16-24i-9)\]
\[-5(7-24i)\]
\[-35i + 126i^2\]
\[-35i - 120\]

We usually write this in standard form:
\[-120 - 35i\]

---

**Powers of i (notice the pattern)**

\[
\begin{array}{ll}
i & i^2 = i \\
i^3 = -1 & i^4 = 1 \\
i^5 = -i & \text{NOTE Every 4\textsuperscript{th} power of i is 1.} \\
i^6 = 1 & \text{SO, a multiple of 4 = 1} \\
i^7 = -i & \text{Strategy: Divide the exponent by 4.} \\
i^8 = -1 & \text{Look at the remainder.} \\
i^9 = i & \text{If remainder = 1 \rightarrow equal to i} \\
i^{10} = -1 & \text{If remainder = 2 \rightarrow equal to -1} \\
i^{11} = -i & \text{If remainder = 3 \rightarrow equal to -i} \\
i^{12} = 1 & \text{If no remainder \rightarrow equal to +1} \\
\end{array}
\]

---

#70) Evaluate \(i^{39}\):

When we divide 39 by 4, we have a remainder of 1. Therefore,  
\[i^{39} = i^1 = i\]

---

#74) Evaluate \(i^{87}\):

When we divide 87 by 4, we have a remainder of 3. Therefore,  
\[i^{87} = i^3 = -i\]

---

#77) Evaluate \(i^{12}\):

We write  
\[i^{12} = i^{16/2} = (i^4)^4 = 1^4 = 1\]
#78) Evaluate $\int^{14}$:

We write

$$\int^{14} = \int^{14^2} = \pi^2 - 1$$

Definition: Given a complex number $a + bi$, its complex conjugate is $a - bi$.

Ex) The complex conjugate of $-2 + 3i$ is what?

$-2 - 3i$

---

Property of Complex Conjugates:

$$(a + bi)(a - bi) = a^2 + b^2$$

Why?

---

Dividing Complex Numbers

Multiply numerator and denominator by the conjugate of the denominator. We say, "rationalize the denominator to remove the $\sqrt{-1}$ (or $i$)"

---

#84) Write the quotient in standard form:

$$\frac{14 + 5i}{3 + 2i}$$

Solution: To rationalize the denominator, we need to multiply the numerator and denominator by the conjugate of the denominator, or $3 - 2i$. We then calculate:

$$\frac{14 + 5i}{3 + 2i} \cdot \frac{3 - 2i}{3 - 2i}$$

$$= \frac{42 - 28i + 15i - 10i^2}{9 - 6i + 6i + 4i^2}$$

$$= \frac{42 - 28i + 15i + 10}{9 - 6i + 6i + 4}$$

$$= \frac{52 - 13i}{13}$$

$$= \frac{52}{13} - \frac{13}{13}i$$

$$= 4 - i$$

The problem is not considered complete until the answer is in this form. All other forms are equivalent, but they are not simplified.

---

#88) Write the quotient in standard form:

$$\frac{-3 + 4i}{2 - i}$$

Solution: To rationalize the denominator, we need to multiply the numerator and denominator by the conjugate of the denominator, or $2 + i$. We then calculate:

$$\frac{-3 + 4i}{2 - i} \cdot \frac{2 + i}{2 + i}$$

$$= \frac{-6 - 3i + 8i + 4i^2}{4 - 2i + 2i + i^2}$$

$$= \frac{-10 + 5i}{5}$$

$$= -2 + i$$
Definition: A quadratic equation is an equation that can be written in the form \( ax^2 + bx + c = 0 \)
where \( a \), \( b \), and \( c \) are real numbers with \( a \neq 0 \).
This format is called standard form.
- called a second-degree equation
- highest power on \( x \) is 2 \((x^2)\)

4 ways to solve:
1. Factoring
2. Square Root Property
3. Completing the Square
4. Quadratic Formula

1) Factoring

Put in standard form.
Must have zero on one side!!

Zero-Factor Property
If \( a \) and \( b \) are complex numbers with \( ab = 0 \),
then \( a = 0 \) or \( b = 0 \) or both.

#10) Solve by using the zero-factor property:

\[ 2x^2 - x - 15 = 0 \]

Solution:

To use the zero-factor property, we need to first factor:
\[ (2x + 5)(x - 3) = 0 \]

Then we set each of these factors to 0:

\[
\begin{align*}
2x + 5 &= 0 \\
2x &= -5 \\
x &= \frac{-5}{2}
\end{align*}
\]

\[
\begin{align*}
x - 3 &= 0 \\
x &= 3
\end{align*}
\]

Therefore, we have the solution \( x = \left\{ \frac{-5}{2}, 3 \right\} \). These are two different answers, not an ordered pair. The order does not matter.

2) Square Root Property

If \( x^2 = k \), then \( x = \sqrt{k} \) or \( x = -\sqrt{k} \).
- Apply a root on both sides of the equation.
- May have imaginary solutions!

#30) Using the square root property, solve

\[ (-2x + 5)^2 = -8 \]

Solution:

Taking the square root of both sides, we have:

\[ -2x + 5 = \pm \sqrt{-8} \]

Using the techniques of the previous section, we can write this as:

\[ -2x + 5 = \pm 2\sqrt{2} \]

Then we subtract the 5 from both sides and divide by the coefficient of \( x \).
\[-2x = -5 \pm 2\sqrt{2} \]
\[x = \frac{-5 \pm 2\sqrt{2}}{2} \]

This is not in standard form, however. We can write this as:
\[x = \frac{-5 \pm \sqrt{2}}{2} \]

3) Completing the Square Method

- Create a Perfect Square on one side of the equation.
- Solve using the Square Root Method.

To solve: \[ax^2 + bx + c = 0 \quad (a \neq 0)\]

1. If \(a \neq 0\), divide both sides of the equation by \(a\).
2. Move the constant term to the right-hand side.
3. Find \(\left(\frac{b}{2a}\right)^2\). Add this to both sides.
4. Factor the left-hand side.
5. Use the square root method to find the solution.

#32) Solve by completing the square:

\[x^2 - 7x + 12 = 0\]

Solution:

We move the constant term to the right-hand side:
\[x^2 - 7x = -12\]
We identify \(b = -7\) and add \(\left(\frac{7}{2}\right)^2\) to both sides:
\[x^2 - 7x + \left(\frac{7}{2}\right)^2 = -12 + \left(\frac{7}{2}\right)^2\]
\[x^2 - 7x + \frac{49}{4} = -12 + \frac{49}{4} - \frac{1}{4}\]

If we have added the correct term, the left-hand side should factor into a squared term:
\[\left(x - \frac{7}{2}\right)^2\]

Now we're able to use the square root method:
\[x - \frac{7}{2} = \pm \frac{\sqrt{4}}{2} = \pm \frac{1}{2}\]
\[x = \frac{7}{2} \pm \frac{1}{2}\]

Now, this is not fully simplified. There are two answers here:
\[x = \frac{7}{2} + \frac{1}{2} = \frac{8}{2} = 4\]
\[x = \frac{7}{2} - \frac{1}{2} = \frac{6}{2} = 3\]

Therefore, \(x = (3, 4)\).

4) Quadratic Formula
Put in standard form:
Must have zero on one side!!

The solutions of \( ax^2 + bx + c = 0 \) (\( a \neq 0 \)) are

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

---

#34) Solve \( \frac{2}{3}x^2 + \frac{1}{4}x = 3 \) with the quadratic formula.

Solution:

We need the equation to be in the form specified by the quadratic equation, so we move the constant to the left-hand side:

\[
\frac{2}{3}x^2 + \frac{1}{4}x - 3 = 0
\]

We could use the quadratic formula with

\( a = \frac{2}{3} \)
\( b = \frac{1}{4} \)
\( c = -3 \)

However, the quadratic formula is relatively complicated as it is, so we do not need the added complication of fractions. We can multiply both sides of the equation by the LCD of the fractions (12) to simplify the calculation:

\[
12\left( \frac{2}{3}x^2 + \frac{1}{4}x - 3 \right) = 12(0)
\]

\[
8x^2 + 3x - 36 = 0
\]

This equation is in the correct form and has:

\( a = 8, b = 3, c = -36 \)

We plug these values into the formula:

\[
x = \frac{-3 \pm \sqrt{3^2 - 4(8)(-36)}}{2(8)}
\]

\[
x = \frac{-3 \pm \sqrt{1151}}{16}
\]

Now, \( 1151 = 9 \times 129 \), so we can write this as:

\[
x = \frac{-3 \pm 3\sqrt{129}}{16}
\]

---

The Discriminant = \( b^2 - 4ac \)

- part of the quadratic formula
- predicts the number and type of solutions

If \( b^2 - 4ac \) equals:

<table>
<thead>
<tr>
<th>Discriminant Value</th>
<th>Number of Solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Positive &amp; Perfect Square</td>
<td>two rational solutions</td>
</tr>
<tr>
<td>Positive &amp; NOT Perfect Square</td>
<td>two real solutions</td>
</tr>
<tr>
<td>Zero</td>
<td>one real solution</td>
</tr>
<tr>
<td>Negative</td>
<td>two imaginary solutions</td>
</tr>
</tbody>
</table>
#76) Use the discriminant to discuss the solutions of

\[ 8x^2 = -14x - 3 \]

Solution:

We begin by writing in standard form:

\[ 8x^2 + 14x + 3 = 0 \]

Therefore, we have \( a = 8, \ b = 14, \ c = 3 \). We then calculate:

\[ b^2 - 4ac = 14^2 - 4(8)(3) = 100 \]

This is a positive number, a perfect square, so we expect that this quadratic should have two rational solutions.

*********

Solving a Cubic Equation

- Use factoring:
  \[ a^3 - b^3 = (a - b)(a^2 + ab + b^2) \]
  \[ a^3 + b^3 = (a + b)(a^2 - ab + b^2) \]

SOAP = "Same – Opposite – Always Positive"

- Use quadratic formula to finish solving

*********

#62) Solve \( x^3 - 27 = 0 \)

Solution:

We notice that

\[ x^3 - 27 = x^3 - (3)^3 \]

so that in the above formulas, \( a = x \) and \( b = 3 \). Therefore,

\[ x^3 - 27 = (x - 3)(x^2 + x(3) + 3^2) \]
\[ = (x - 3)(x^2 + 3x + 9) = 0 \]

Now, by the zero-factor property, we can set each of these factors equal to 0 to obtain our solutions:

\[ x - 3 = 0 \]
\[ x = 3 \]

This is just one of three solutions! We also have to set the second factor equal to 0:

\[ x^2 + 3x + 9 = 0 \]

We can solve this using any of the above techniques. Because the quadratic formula always works, we will probably choose it. In any case, we obtain

\[ x = \frac{-3 \pm 3\sqrt{3}}{2} \]

as two more solutions. Therefore, the solution set is:

\[ x = \left\{ 3, \frac{-3 + 3\sqrt{3}}{2}, \frac{-3 - 3\sqrt{3}}{2} \right\} \]
Applications and Modeling with Quadratic Equations, 1.5

Geometry Problems

#22) Width of a Flower Border

A landscape architect has included a rectangular flower bed measuring 9 ft by 5 ft in her plans for a new building. She wants to use 2 colors of flowers in the bed, one in the center and the other for a border of the same width on all four sides. If she has enough plants to cover 24 ft² for the border, how wide can the border be?

Solution:

We know that the flower bed has a total area of 9 x 5 = 45 ft². The border has an area of 24 ft² so the center section has area

\[ 45 - 24 = 21 \text{ ft}^2 \]

Call the width of the border \( x \) ft. Then length of the center area is

\[ L = 9 - 2x \]

and the width if the center area is

\[ W = 5 - 2x \]

Therefore, we can write

\[ A = LW = 21 = (9 - 2x)(5 - 2x) \]

This is just a quadratic equation. We can use any of the methods in the previous section to solve it. In any case, we obtain the two solutions \( x = \{1, 5 \} \). Quadratics do usually have two solutions. In this case, however, only one of these solutions makes good physical sense. For instance, if we have a border that is 8 feet wide and the flower bed is only 9 feet wide, the border area is larger than the total area, which does not make sense.

Therefore, the border is 1 ft wide, yielding a center area of dimensions 7 by 3.

#24) Volume of a Box

A rectangular piece of metal has a length that is twice the width. Squares with sides 4 in. long are cut from four corners, and the flaps are folded upward to form an open box.

If the volume of the box is 1536 in³, what were the original dimensions of the piece of metal?

Solution:

It is good to cut a piece of paper and fold it in the manner described. We know that the height (\( h \)) of the resulting box is 4 inches. Since the width (\( w \)) of the original piece of metal is unknown, call it \( x \).

Now we know that the length (\( l \)) of the original piece of metal is twice the width, or 2\( x \). Therefore, the dimensions of the box are:

\[ h = 4 \]

\[ w = x - 2(4) = x - 8 \]

\[ l = 2x - 2(4) = 2x - 8 \]

Now, we know that the volume of the box is given by

\[ V = 1536 = LWH \]

which gives us the equation

\[ 1536 = (2x - 8)(x - 8) \]

We can use any of the techniques of the previous section to give us the solution. We solve and obtain

\[ x = 20 \]

However, because we are dealing with physical quantities, \( x \) must necessarily be positive. Therefore,

\[ w = x = 20 \]

\[ l = 2x = 40 \]

Therefore, our original sheet of metal has dimensions 20 in x 40 in.

#32) Height of a Kite

A kite is flying on 50 ft of string. Its vertical distance from the ground is 10 ft more than its horizontal distance from the person flying it. Assuming that the string is being held at ground level, find its horizontal distance from the person and its vertical distance from the ground.

We can view this as a right triangle with hypotenuse and an unknown height (\( h \)) and an unknown width (\( w \)). Call the width \( w \). Then we know that the height above the ground is 10 more than this, i.e.

\[ h = w + 10 = x + 10 \]

Now, using the Pythagorean Theorem, we can write

\[ \text{(height)}^2 + \text{(width)}^2 = \text{(hypotenuse)}^2 \]

\[ (x + 10)^2 + x^2 = 50^2 \]
This is just a quadratic, so we can solve it using any of the techniques of the last section. We obtain \( x = [-40, 30] \). Obviously, however, widths and heights cannot be negative. Therefore, we include that \( x \), the width, is 30, and the height is 10 more than this, or 40.

Therefore, the vertical distance from the ground is 30 ft. and the horizontal distance is 40 ft.

---

### 4.3) Height of a Projectile

An astronaut on the moon throws a baseball upward. The astronaut is 6 ft. 6 in. tall and the initial velocity of the ball is 30 ft/sec. The height \( s \) of the ball in feet is given by the equation: \( s = -2.7t^2 + 30t + 6.5 \) where \( t \) is the number of seconds after the ball was thrown.

a. After how many seconds is the ball 12 ft above the moon’s surface? (Round to the nearest hundredth.)

Solution:

We set

\[ 12 = -2.7t^2 + 30t + 6.5 \]

and can solve this using any of the methods of the previous section. Because we are dealing with decimals, it will likely be easiest to use the quadratic formula. We obtain

\[ t = 0.19, 10.92 \]

Does it make sense that there would be two solutions?

b. How many seconds will it take for the ball to return to the surface of the moon? (Round to the nearest hundredth.)

When the ball is at the surface of the moon, its height is 0, so we solve

\[ 0 = -2.7t^2 + 30t + 6.5 \]

Solving this equation, we obtain \( t = -0.21, 11.32 \). In this case, only one of these solutions makes sense. Therefore, the ball returns to the surface of the moon at around 11.32 seconds.

---

### 4.7) Carbon Monoxide Exposure

High concentrations of carbon monoxide (CO) can cause coma and death. The time required for a person to reach a level capable of causing a coma can be approximated by the quadratic model

\[ T = 0.0002x^2 - 0.31x + 127.9 \]

where \( T \) is the exposure time in hours necessary to reach this level and 500 \( \leq x \leq 800 \) is the amount of carbon monoxide present in the air in parts per million (ppm).

a. What is the exposure time when \( x = 800 \) ppm?

We calculate

\[ T = 0.0002(800)^2 - 0.31(800) + 127.9 \]

\[ T = 10.3 \]

Therefore, the exposure time is 10.3 hours.

b. Estimate the concentration of CO necessary to produce a coma in 4 hours.

We seek to solve

\[ 4 = 0.0002x^2 - 0.31x + 127.9 \]

We can use any of the methods of the previous section. We obtain

\[ x = 722.18, 857.82 \]

However, only one of these solutions is between 500 and 800, as specified by the problem. Therefore, the concentration is 722 ppm.
Powers of $i$

$i = \sqrt{-1}$, so it is straightforward that $i^2 = -1$. What about $i^3$? We can just multiply $i$ by $i^2$: $i^3 = i(i^2) = i(-1) = -i$. And how about $i^4$? $i^4 = i^2i = -i(i) = -i^2 = -(-1) = 1$.

After working with the powers of $i$ for a while, you might start to notice a pattern:

<table>
<thead>
<tr>
<th>$i^1$</th>
<th>$i^2$</th>
<th>$i^3$</th>
<th>$i^4$</th>
<th>$i^5$</th>
<th>$i^6$</th>
<th>$i^7$</th>
<th>$i^8$</th>
<th>$i^9$</th>
<th>$i^{10}$</th>
<th>$i^{11}$</th>
<th>$i^{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i$</td>
<td>$-1$</td>
<td>$-i$</td>
<td>$1$</td>
<td>$i$</td>
<td>$-1$</td>
<td>$-i$</td>
<td>$1$</td>
<td>$i$</td>
<td>$-1$</td>
<td>$-i$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

What information can we get from this table? The powers of $i$ repeat in a cycle. When $i$ is raised to a power that is divisible by 4, the answer is 1. For exponents not divisible by 4, we can look at the remainder when that exponent is divided by 4. If the remainder is 1, the answer will be $i$. If it is 2, the answer will be -1. If it is 3, it will be $-i$.

How can we use a calculator to help us? If we take the exponent and divide by 4, let’s look at what is after the decimal point. The only possibility (for integer powers greater than 0) is .25, .5, .75, and nothing. Since .25 = $\frac{1}{4}$, this represents a remainder of 1 when divided by 4. And so .5 represents a remainder of 2 when divided by 4, and .75 a remainder of 3.

Example: What is $i^{203}$?

$\frac{203}{4} = 50.75$, and we only look at what is after the decimal point. Since it is .75, the remainder when 203 is divided by 4 is 3. Thus $i^{203} = i^3 = -i$.

Example: What is $i^{2010}$?

$\frac{2010}{4} = 502.5$, so the remainder when 2010 is divided by 4 is 2. So $i^{2010} = i^{2008+2} = i^{2008}i^2 = i^2 = -1$.

In summary, to find $i^{\text{number}}$, take the number and divide by 4. If:

- $\frac{\text{number}}{4} = \text{xxxx}.25$, then the answer is $i$. (the xxxx represents the part before the decimal)
- $\frac{\text{number}}{4} = \text{xxxx}.5$, then the answer is -1.
- $\frac{\text{number}}{4} = \text{xxxx}.75$, then the answer is $-i$.
- $\frac{\text{number}}{4} = \text{xxxx}.00$, then the answer is 1.
Now let’s consider negative exponents. Recall that \( a^{-r} = \frac{1}{a^r} \). The same thing works for \( i \).

Example: Simplify \( i^{17} \).
In order to work with this, we need to take care of the negative first. So let’s move \( i^{17} \) to the denominator: \( \frac{1}{i^{17}} \). Now we know when 17 is divided by 4, the remainder is 1. So \( i^{17} = i^1 = i \), so that means \( \frac{1}{i^{17}} = \frac{1}{i} \). Now remember that \( i \) is the square root of -1, so in the expression \( \frac{1}{i} \) there is a radical in the denominator (this is like simplifying \( \frac{1}{\sqrt{2}} \)). So we multiply by a strategic 1 and get \( \frac{1}{i} \left( \frac{i}{i} \right) = \frac{i}{i^2} = \frac{i}{-1} = -i \). Thus \( i^{-17} = -i \).

Example: Simplify \( i^{3607} \).
We know that \( i^{3607} = \frac{1}{i^{3607}} \). Dividing 3607 by 4, we get 901.75. The .75 tells us that there is a remainder of 3 when 3607 is divided by 4, so \( i^{3607} = i^3 = -i \). So we have \( \frac{1}{i^{3607}} = \frac{1}{-i} \). Now we need to take care of the square root in the denominator, so we have \( \frac{1}{-i} \left( \frac{-i}{-i} \right) = \frac{-i}{i^2} = \frac{-i}{-1} = i \).

Example: Simplify \( \frac{i^{92} + i^{93}}{i^{2000} - i^{37}} \).
Separately we will have to evaluate each of these. Please check that \( i^{92} = 1 \) (no remainder upon dividing by 4), \( i^{93} = i \) (multiply \( i^{92} \) by \( i \)), \( i^{2000} = 1 \) (no remainder upon dividing by 4), and \( i^{37} = i \) (remainder of 1 upon dividing by 4). We carefully place all of these back in the expression: \( \frac{1+i}{1-i} \). Now we must rationalize the denominator by multiplying by the conjugate.

Remember: The conjugate of \( a + bi \) is \( a - bi \). Thus the conjugate of \( 1-i \) is \( 1+i \). So to not change the value of the expression but get the radical out of the denominator, we multiply by a form of 1: \( \frac{1+i}{1-i} \left( \frac{1+i}{1+i} \right) = \frac{(1+i)(1+i)}{(1-i)(1+i)} = \frac{1+i+i^2}{1+1-i^2} = \frac{1+2i-1}{2} = \frac{2i}{2} = i \).
Completing the Square

Everyone loves the square root method, because it is easy. To solve \( x^2 = 4 \), we take the square root of both sides (and not forgetting the plus/minus) we get the solution of \( x = \pm 2 \). But what if there is a linear term (a term with \( x \) in it). Completing the square allows us to use the square root method for any quadratic equation. We make the left-hand side a perfect square trinomial, something that looks like \((x + a)^2\). Taking the square root of that is easy.

From the lecture notes, we know the steps for completing the square (i.e., solving \( ax^2 + bx + c = 0 \)) are:

1. If \( a \neq 1 \), divide both sides of the equation by \( a \).
2. Move the constant to the right-hand side.
3. Find \( \left(\frac{1}{2}b\right)^2 \), where \( b \) is the coefficient of the \( x \) (after Step 1). Add this to both sides.
4. Factor the left-hand side.
5. Use the square root method to find the solution.

Let’s go through some examples using these 5 steps.

Example: Solve \( x^2 - 4x + 2 = 0 \).

<table>
<thead>
<tr>
<th>Step</th>
<th>Equation</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( x^2 - 4x + 2 = 0 )</td>
<td>( a = 1 ) so nothing needs to be done.</td>
</tr>
<tr>
<td>2</td>
<td>( x^2 - 4x = -2 )</td>
<td>Move the constant 2 to the other side.</td>
</tr>
<tr>
<td>3</td>
<td>( x^2 - 4x + 4 = -2 + 4 )</td>
<td>( b = -4 ), ( \frac{1}{2}b = -2 ), ( \left(\frac{1}{2}b\right)^2 = (-2)^2 = 4 )</td>
</tr>
<tr>
<td>4</td>
<td>( x^2 - 4x + 4 = 2 )</td>
<td>Simplify, and factor. Notice that ( x^2 - 4x + 4 = (x - 2)(x - 2) = (x - 2)^2 )</td>
</tr>
<tr>
<td>5</td>
<td>( x - 2 = \pm \sqrt{2} ) ( x = 2 \pm \sqrt{2} )</td>
<td>Take plus/minus the square root of both sides, and solve for ( x ).</td>
</tr>
</tbody>
</table>
Example: Solve \(-2x^2 + 8x - 9 = 0\).

<table>
<thead>
<tr>
<th>Step</th>
<th>Equation</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(-2x^2 + 8x - 9 = 0) [-\frac{2}{2} + \frac{8}{2} + \frac{-9}{2} = 0] [x^2 - 4x + \frac{9}{2} = 0]</td>
<td>Remember to divide each expression on the left by -2, not just the first one.</td>
</tr>
<tr>
<td>2</td>
<td>(x^2 - 4x = \frac{9}{2})</td>
<td>Move (\frac{9}{2}) to the other side.</td>
</tr>
<tr>
<td>3</td>
<td>(x^2 - 4x + 4 = -\frac{9}{2} + 4)</td>
<td>(b = -4), (\left(\frac{1}{2} b\right)^2 = \left(-2\right)^2 = 4), add to both sides.</td>
</tr>
<tr>
<td>4</td>
<td>(x^2 - 4x + 4 = -\frac{9}{2} + \frac{8}{2}) [(x - 2)^2 = -\frac{1}{2}]</td>
<td>Simplify, and factor. Remember the left should factor into something squared.</td>
</tr>
<tr>
<td>5</td>
<td>((x - 2)^2 = -\frac{1}{2}) [x - 2 = \pm \sqrt{-\frac{1}{2}}] [x - 2 = \pm \frac{i}{\sqrt{2}}] [x - 2 = \pm \frac{i}{\sqrt{2}} \left(\frac{\sqrt{2}}{\sqrt{2}}\right)] [x = 2 \pm \frac{\sqrt{5}}{2}i]</td>
<td>Take the square root. Don’t forget the plus/minus. [\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}]. [i = \sqrt{-1}]. Rationalize the denominator (multiply by strategic 1). Finish solving.</td>
</tr>
</tbody>
</table>
Textbook Section 1.6, Other Types of Equations

Objectives

- The student will be able to solve problems involving work
- The student will be able to solve equations with rational expressions
- The student will be able to solve equations with radicals
- The student will be able to solve equations in quadratic form

Key Concepts

Work Rate Problems

- rate \times time = part of job accomplished
- \( r = \frac{1}{t} \)

Rational Equation

- has a rational expression for one or more terms
- Restrictions: solutions cannot make the denominator zero
- Strategy: factor all denominators, multiply both sides by the LCD

Solving Equations with Radicals

1. Isolate a radical.
2. Raise each side to an appropriate power.

If the equation still contains a radical, repeat steps 1 & 2.

3. Solve the equation.
4. Check answers.
   Extraneous solutions may appear!!

Equation in quadratic form

1. \( ax^2 + bx + c = 0, (a \neq 0) \)
   where \( u \) is some algebraic expression.
2. Strategy: called \( u \)-substitution

Handouts

Some of the files you are about to view/download are PDF files. If you do not have Adobe Acrobat installed on your system, you can download the free Adobe Acrobat Reader at http://www.adobe.com/products/acrobat/alternatel.html

- Substitution to Solve Quadratics Handout

Online Homework and Quiz Assignments

After reviewing the Key Concepts and Handouts, log into MyLabsPlus and begin your homework and quiz for this week. Go to www.ucl.mylabsplus.com and begin working on your assignments.
Rational Equations

Definition: A Rational Equation is an equation that has a rational expression for one or more terms.

- Restrictions: solutions cannot make the denominator 0
- Strategy: multiply both sides by the LCD

\[ \frac{3}{x^2-4} + \frac{1}{x^2-2} = \frac{12}{x^2-4} \]

Solution:

We can write

\[ x^2-4 = (x-2)(x+2) \]

so the LCD of these fractions is \((x-2)(x+2)\). We multiply both sides of the equation by this factor:

\[ \frac{3}{x^2-2} - \frac{1}{x^2-2} = \frac{12}{x^2-4} \]

We distribute and cancel where we can:

\[ 3(x+2) - 1(x-2) = 12 \]

This is now a linear equation, so we solve:

\[ 3x + 6 + x - 2 = 12 \]

\[ 4x + 4 = 12 \]

\[ x = 2 \]

However, we note that plugging in 2 makes some of the denominators equal to 0. Therefore, the solution is \( \emptyset \).

\[ \frac{3}{2x-1} + \frac{1}{(2x-1)^2} \]

Solution:

We multiply both sides of the equation by the LCD, which is \((2x-1)^2\):

\[ 2(2x-1)^2 = 3(2x-1) - 1 \]

This is now a quadratic, so we can solve it using any of the methods of the previous chapter. We obtain \( x = \left\{ \frac{3}{2}, 1 \right\} \). Neither of these solutions makes denominators 0.

Work Rate Problems

If a job can be done in \( t \) units of time:

\[ \text{rate} \times \text{time} = \text{portion of job completed} \]

If 1 whole job is accomplished:

rate \times time = 1 OR

\[ rt = 1 \]

\[ r = \frac{1}{t} \]

Joe can paint a house in 3 hours.
Sam can paint the same house in 5 hours.
How long does it take them to work together?

Solution:

Joe’s rate is 1/3 of a house per hour. Sam’s rate is 1/5 of a house per hour. Therefore, if Joe works for \( t \) hours, he has painted \( \frac{1}{3} t \) of a house in that time. If Sam works for the same amount of time, Sam has painted \( \frac{1}{5} t \).

Therefore, working together to paint one house, we have
\[
\frac{1}{3} + \frac{1}{5} = 1 \\
5t + 3t = 15 \\
8t = 15 \\
t = \frac{15}{8}
\]

Therefore, it takes 1.975 hours for them to paint the house together.

---

**Equations with Radicals**

Use the Power Property to solve:

If \(P\) and \(Q\) are algebra expressions, every solution of \(P = Q\) is a solution of \(P^n = Q^n\), for any positive integer \(n\).

To solve a radical equation:

1. Isolate the radical.
2. Raise each side to a power.
3. Solve the equation.
4. Check answers.

Extraneous solutions may appear!!

Repeat 1 and 2 if necessary to remove all radicals.

---

\#4a) \(\sqrt{2x} - x + 4 = 0\)

We first isolate the radical and square both sides:

\[
\begin{align*}
\sqrt{2x} - x + 4 &= 0 \\
\sqrt{2x} &= x - 4 \\
(\sqrt{2x})^2 &= (x - 4)^2 \\
2x &= x^2 - 8x + 16 \\
0 &= x^2 - 10x + 16
\end{align*}
\]

This can be factored easily:

\[
0 = (x - 2)(x - 8)
\]

So we have solutions \(x = 2, 8\). However, we need to check them. We plug in:

\[
\sqrt{2(2)} - 2 + 4 = 4 \neq 0
\]

So \(x = 2\) is an extraneous solution. However, when we plug in \(x = 8\):

\[
\sqrt{2(8)} - 8 + 4 = 0
\]

Therefore, our solution set is \(x = 8\). 2 is not a solution.

---

\#4b) \(\sqrt{4x + 1} - \sqrt{x - 1} = 2\)

We begin by isolating one of the radicals and squaring both sides:

\[
\begin{align*}
\sqrt{4x + 1} &= 2 + \sqrt{x - 1} \\
(\sqrt{4x + 1})^2 &= (2 + \sqrt{x - 1})^2 \\
The \text{left-hand side is easy, but we have to remember to FOIL the right-hand side:} \\
4x + 1 &= 4 + 4\sqrt{x - 1} + (x - 1) \\
We \text{collect terms and isolate the remaining radical:} \\
3x - 2 &= 4\sqrt{x - 1} \\
We \text{square both sides again:} \\
3x - 2 &= 4\sqrt{x - 1} \\
(3x - 2)^2 &= (4\sqrt{x - 1})^2
\end{align*}
\]
9x^2 - 12x + 4 = 16(x^2 + 1)

We collect terms:
9x^2 - 12x + 4 = 16x^2 - 16
9x^2 - 28x + 20 = 0

We can use any of the methods of the previous section to solve this quadratic. We obtain:
\[ x = \frac{10}{9} \]

In fact, when we plug in \( x = 2 \), we obtain:
\[ \sqrt{4(2)} + 1 - \sqrt{2} - 1 + \sqrt{2} - \sqrt{1} - 3 - 1 = 2 \]

We can show as well that \( x = \frac{10}{9} \) is a solution. Therefore, the solution set is:
\[ x = \frac{10}{9} \]

---

\#58 \( \sqrt{2x} = \sqrt{5x + 2} \)

We begin by raising both sides of the equation to the third power:
\[
\left( \sqrt{2x} \right)^3 = \left( \sqrt{5x + 2} \right)^3
\]
\[ 2x = 5x + 2 \]
\[ -3x = 2 \]
\[ x = -\frac{2}{3} \]

---

Equations in quadratic form

Definition: An equation in quadratic form can be written
\[ au^2 + bu + c = 0, a \neq 0 \]

where \( u \) is some algebraic expression.

Strategy: called \( u \)-substitution

---

\#72 \( 3x^4 + 10x^2 - 25 = 0 \)

If we write \( u = x^2 \) then we know that \( x^4 = (x^2)^2 = u^2 \). Therefore, the equation becomes
\[ 3u^2 + 10u - 25 = 0 \]

We can solve this using any of the techniques of the previous section to obtain \( u = 5, -\frac{5}{3} \). However, we were asked to solve for \( x \). Therefore, we write the two new equations
\[ u = 5 \Rightarrow x^2 = 5 \]
\[ x = \pm \sqrt{5} \]
\[ u = -\frac{5}{3} \Rightarrow x^2 = -\frac{5}{3} \]
\[ x = \pm \sqrt{-\frac{5}{3}} \]

Therefore, there are four solutions, \( x = \pm \sqrt{5}, \pm \sqrt{-\frac{5}{3}} \)

---

\#78 \( (2x-1)^{\frac{2}{3}} + 2(2x-1)^{\frac{1}{3}} - 3 = 0 \)

If we write
\[ u = (2x-1)^{\frac{1}{3}} \]
then we know that
\[ u^2 = (2x-1)^{\frac{2}{3}} \]

We substitute this into the original equation to obtain:
\( y^2 + 2u - 3 = 0 \)
\( (u + 3)(u - 1) = 0 \)

which tells us that \( u = 1, -3 \). We then solve

\[
\begin{align*}
  u &= (2x - 1)^\frac{1}{2} = 1 \\
  2x - 1 &= 1^2 = 1 \\
  2x &= 2 \\
  x &= 1 
\end{align*}
\]

Therefore, this is one of the solutions.

\[
\begin{align*}
  u &= (2x - 1)^\frac{1}{2} = -3 \\
  2x - 1 &= (-3)^2 = 9 \\
  2x &= 26 \\
  x &= -13 
\end{align*}
\]

Therefore, we have the two solutions \( x = 1, -13 \). We can plug these back in to check:

\[
\begin{align*}
  (2(1) - 1)^\frac{1}{2} + 2(2(1) - 1)^\frac{1}{2} - 3 \\
  = 1^\frac{1}{2} + 2(1)^\frac{1}{2} - 3 \\
  = 1 + 2 - 3 = 0 \\
  \end{align*}
\]

\[
\begin{align*}
  (2(-13) - 1)^\frac{1}{2} + 2(2(-13) - 1)^\frac{1}{2} - 3 \\
  = (-27)^\frac{1}{2} + 2(-27)^\frac{1}{2} - 3 \\
  = -3 + 2(-3) - 3 \\
  = 9 - 6 - 3 = 0 
\end{align*}
\]

#94) \( x^\frac{1}{2} + y^\frac{1}{2} = a^\frac{1}{2} \) solve for \( y \)

We write:

\[
y^\frac{1}{2} = a^\frac{1}{2} - x^\frac{1}{2}
\]

We can then raise both sides to the \( \frac{3}{2} \) power:

\[
\left(y^\frac{1}{2}\right)^{\frac{3}{2}} = y = \pm \left(a^\frac{1}{2} - x^\frac{1}{2}\right)^{\frac{3}{2}}
\]

We are including the \( \pm \) because we are taking a half root. This is like taking the square root; in fact, we could do this in two steps:

\[
y^\frac{3}{2} = \pm \sqrt[3]{a^\frac{1}{2} - x^\frac{1}{2}}
\]

\[
y = \pm \left(\sqrt[3]{a^\frac{1}{2} - x^\frac{1}{2}}\right)^{\frac{3}{2}}
\]

These two answers are equivalent.
Using substitution to solve disguised quadratics

Substitution is one of the most powerful tools in mathematics. It helps us see something complicated in a familiar manner. Take quadratic equations: \( ax^2 + bx + c = 0 \). Up until now, we have been solving them in terms of \( x \). We can actually solve any equation with three terms where one expression has the power doubled of the other (and the third being a constant). If we have the equation \( 4x^4 + 5x^2 + 6 = 0 \), we can let \( y = x^2 \) to obtain the equation \( 4y^2 + 5y + 6 = 0 \).

The first equation is actually a quadratic in \( x^2 \): \( 4(x^2)^2 + 5(x^2) + 6 = 0 \). All we do is a simple substitution and the equation becomes more manageable.

*** We usually substitute the middle variable raised to its current power. If the first term has double the exponent, we are in good shape.

*** DON’T forget to substitute back in for your original variable when you finish solving!

**Example:** Solve \( 6x^6 - 217x^3 + 27 = 0 \).

This looks very hard. But once we see the 6 is double the 3, this might turn out to be a quadratic. An easy test is to let ‘u’ or some variable equal the middle variable (raised to its current power). Let’s let \( u = x^3 \). Then our equation is transformed into \( 8u^2 - 217u + 27 = 0 \). We can use any method to solve this now: completing the square, quadratic formula, etc. Let’s factor:

\[
(8u - 1)(u - 27) = 0.
\]

So we have \( u = \frac{1}{8}, \ u = 27 \). Guaranteed these will be choices on your test, but they will be wrong. Why? Because they are in terms of \( u \), not \( x \). Now we substitute back for \( x \): \( x^3 = \frac{1}{8}, \ x^3 = 27 \). Now we take the cube root of both sides: \( x = \left(\frac{1}{8}\right)^{\frac{1}{3}} = \frac{1}{2}, \) and

\[
x = 27^{\frac{1}{3}} = 3.
\]

Our final answers are 0.5 and 3. (Why is the answer not plus/minus? Because we are taking an odd root, and when we raise things to an odd power they keep their sign.)

**Example:** Solve \( 4x^\frac{4}{3} - 13x^\frac{2}{3} + 36 = 0 \).

Notice how \( \frac{4}{3} = 2 \left(\frac{2}{3}\right) \), so one exponent is double the other. Let’s try a substitution for the middle variable: \( u = x^{\frac{2}{3}} \). The equation is really \( \left(x^{\frac{2}{3}}\right)^2 - 13\left(x^{\frac{2}{3}}\right) + 36 = 0 \), so using our substitution we have \( u^2 - 13u + 36 = 0 \). This is easy to factor: \( (u - 4)(u - 9) = 0 \). But 4 and 9 are not the answers! We have \( u = 4, \ u = 9 \), but \( x^{\frac{2}{3}} = 4, \ x^{\frac{2}{3}} = 9 \). Now let’s solve each equation.
\[ x^2 = 4, \quad x = \pm (4)^{\frac{3}{2}} = \pm \left( 4^{\frac{1}{2}} \right)^3 = \pm 2^3 = \pm 8. \] The second set of solutions is
\[ x^2 = 9, \quad x = \pm (9)^{\frac{3}{2}} = \pm \left( 9^{\frac{1}{2}} \right)^3 = \pm 3^3 = \pm 27. \] So the solution set is \( \{ -8, 8, -27, 27 \} \).

(Again please note that there are 4 solutions to this equation, but only two for the first example. That’s because we had to take an even root, so our solutions could be positive or negative, since when we take an even power they are always positive.)

Example: Solve \( x^{-\frac{2}{5}} - 3x^{-\frac{1}{5}} + 2 = 0 \).

Notice the first exponent is double the second, so we let \( u = x^{-\frac{1}{5}} \), and our equation becomes \( u^2 - 3u + 2 = 0 \). This factors easily into \((u-1)(u-2) = 0\), so we have \( u = 1, \ u = 2 \). We are not done yet, because \( x^{-\frac{1}{5}} = 1, \ x^{-\frac{1}{5}} = 2 \). In each equation we raise both sides to the -5 power, because that is what will give us \( x^1 \) (remember the rules for exponents, when we raise a power to a power we multiply, so we need the multiplicative inverse of \( -\frac{1}{5} \)).

\[
\begin{align*}
  x^{-\frac{1}{5}} &= 1 & & x^{-\frac{1}{5}} &= 2 \\
  \left( x^{-\frac{1}{5}} \right)^{-5} &= (1)^{-5} & & \left( x^{-\frac{1}{5}} \right)^{-5} &= (2)^{-5} \\
  x &= 1 & & x &= \frac{1}{2^5} = \frac{1}{32}
\end{align*}
\]

Our solution set is \( \left\{ 1, \frac{1}{32} \right\} \).
Test Scheduling and Taking the Test

Scheduling a Testing Appointment

- In order to take a test, you must schedule a reservation time.
- Without a reservation, you will not be admitted to the testing room or allowed to take a make-up exam.
- Please recognize that unless you receive a confirmation number and/or confirmation email, you are not registered for your test!
- Registration closes at 8:00pm before the first day of testing. Test scheduling open and close dates are listed online in the test scheduling environment.

If you fail to schedule a test by the deadline, you will receive a zero for that exam. The final exam is the only exception to this policy.

To Make a Reservation for a Testing Session

- Log in to MyLabsPlus through the website www.ucf.mylabsplus.com
- Click on your course.
- Click the “Test Scheduling” link on the left-hand menu bar.
- Enter your NID and last name (first letter capitalized).
- Once you’ve successfully logged into the reservation system, click on a date to create a reservation. The testing dates for each test are listed in the syllabus.
- After deciding on the best available date and time, confirm your email address and complete your reservation.
- Confirm your reservation by checking your Knights email account for the confirmation email.
- You may log into the test scheduling system to confirm your testing appointment. Provided test scheduling is still open, you can also change your reservation.
- Please be aware that there are select dates when the test scheduling will be open to students. These dates will be announced and are posted on the test scheduling website.

Test Taking

To be admitted to the testing session, you must have three things:

1. A testing reservation
2. Your UCF ID (no other ID will be accepted)
3. A new 8.5"x11" Blue Book (smaller Blue Books are unacceptable)

It is also highly recommended that you bring the following as well

- Pen or pencil
- TI-30XA calculator (no other calculator is permitted)
- Knowledge of your MyLabsPlus login and password

If it is necessary to retrieve login credentials subsequent to the student’s admittance to the testing room, the test will begin first, and that student will lose some testing time.

Textbook Section 1.7
Inequalities

Objectives

- The student will be able to use interval notation
- The student will be able to solve linear inequalities
- The student will be able to solve problems involving revenue and cost
- The student will be able to solve polynomial inequalities
- The student will be able to solve rational inequalities

Key Concepts

Linear Inequality

- A linear inequality is of the form \( ax + b < 0 \), where \( a \neq 0 \)
- We can solve the inequality much like we would solve an equation
- If we multiply or divide by a negative number, we must reverse the direction of the inequality sign

Quadratic Inequality

A quadratic inequality is of the form: 
\[ ax^2 + bx + c < 0 \]

where \( a \neq 0 \)

To find the solutions, or solve, a quadratic equation,

1. Replace \( < \) with \( = \) to create a related equation
2. Solve the related equation to find the critical points which are the solutions to the related equation. Notice the intervals created by critical points.
3. Test a value from each interval to decide which intervals are included in the solution set

**Rational Inequality**

Rewrite: single fraction on one side, zero on other side

Find the critical points

Find \( x \)-values that make the numerator zero

Find \( x \)-values that make the denominator zero

Test a value from each interval to decide which intervals are included in the solution set

---

**Textbook Section 1.8**

**Absolute Value Equations and Inequalities**

**Objectives**

- The student will be able to solve absolute value equations
- The student will be able to solve absolute value inequalities
- The student will be able to solve modeling problems involving absolute value

**Key Concepts**

**Absolute Value**

For \( |a| = b \) and \( |a| = |b| \), set \( a = b \) and \( a = -b \).

For \( |a| < b \), set \( a < b \) and \( a > -b \). \((-b < a < b)\)

For \( |a| > b \), set \( a < -b \) or \( a > b \). \((\text{Do not write} -b > a > b)\)

---

**Handouts**

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- Linear Inequalities and Interval Notation Handout
- Quadratic Inequalities Handout
- Rational Inequalities Handout
- Absolute Value Handout

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**Online Homework and Quiz Assignments**

After reviewing the Key Concepts, log into MyLabPlus and begin your homework and quiz for this week. Then go to [www.ucf.mylabsplus.com](http://www.ucf.mylabsplus.com) and begin working on your assignments.

---

**Practice Test Assignments**

Although these are not identical to the actual test, the majority of the questions on the test will come from or be very similar to the practice tests.

---

**Reminders**

This week, you will be scheduling a testing appointment. Please be sure to confirm that you have an appointment by clicking on check reservation when you have completed the process. There are several helpful handouts for the course material this week. Don’t forget to complete the practice tests.
Inequalities, 1.7

Definition: A linear inequality in one variable is an inequality that can be written in the form \(ax + b > 0\), where \(a\) and \(b\) are real numbers with \(a \neq 0\), (or \(<\), \(\leq\), \(\geq\), \(\geq\))

- Solution is a set of numbers.
  "a section of the number line versus just a point"

Interval Notation

- or \(\geq\) round parenthesis () endpoint is NOT included.
- \(\leq\) or \(\geq\) square bracket [] endpoint IS included.

\[ x < -6 \]

To solve:

1. Solve like an equation.
   - Add/Subtract on both sides
     - If \(a < b\), then \(a + c < b + c\).
   - Multiply/Divide both sides by a positive number, \(c\)
     - If \(a < b\), then \(ac < bc\).

2. If we multiply or divide by a negative number, then we must reverse the inequality.
   - Multiply/Divide both sides by a negative number, \(c\)
   - If \(a < b\) then \(ac > bc\).

---

\#20) \(2 - 4x + 5(x - 1) < -6(x - 2)\)

We calculate:

\[
\begin{align*}
2 - 4x + 5x - 5 &< -6x + 12 \\
x - 3 &< -6x + 12 \\
7x &< 15 \\
x &< \frac{15}{7}
\end{align*}
\]

---

\#34) \(1 \leq \frac{4x - 5}{2} < 9\)

We multiply all parts of the equation by \(-2\) to clear the fraction:

\[
\begin{align*}
1(-2) &\geq 4x - 5 > 9(-2) \\
-2 &\geq 4x - 5 > -18 \\
-2 + 5 &\geq 4x > -18 + 5 \\
3 &\geq 4x > -13 \\
\frac{3}{4} &\geq x > -\frac{13}{4}
\end{align*}
\]

Normally, though, this is written in the opposite order:

\[
-\frac{13}{4} < x \leq \frac{3}{4}
\]
Revenue = money gained from the sale of a product, \( R(x) \)
Cost = money needed to produce a product, \( C(x) \)

**Break-Even Point:** the number of products, \( x \), produced and sold so that \( R(x) = C(x) \)

**NOTE:** \( R(x) > C(x) \) causes a profit
\( R(x) < C(x) \) causes a loss

---

36) Find all the intervals where the product will at least break even.

\[ R(x) \geq C(x) \]

The cost to produce \( x \) units of baseball caps is \( C = 100x + 6000 \), while the revenue is \( R = 500x \).

We write:

\[
500x \geq 100x + 6000
\]

\[
400x \geq 6000
\]

\[
x \geq \frac{6000}{400} = 15
\]

Therefore, the product will at least break even when 15 are produced.

---

**Quadratic Inequalities**

Definition: A quadratic inequality can be written in the form \( ax^2 + bx + c < 0 \) for real numbers \( a, b, c (a \neq 0) \) (or \( \leq, \geq, \leq \leq \))

- solution is a set of numbers

To solve:

1. Replace inequality with \( = \). Solve the quadratic equation. Solution will be the critical points.
2. Find the intervals created by critical points.
3. Test a value from each interval to decide which intervals are included in the solution set.

---

40) \( x^2 - 7x + 10 > 0 \)

We replace the inequality symbol with \( = \) and seek critical points:

\[
x^2 - 7x + 10 = 0
\]

\[
(x-2)(x-5) = 0
\]

so that we have the critical points \( x = 2, 5 \).

Therefore, we have the three intervals of the real number line:

<table>
<thead>
<tr>
<th>Interval</th>
<th>Test Point</th>
<th>Is the inequality true at this point?</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-\infty, 2])</td>
<td>(x = 0)</td>
<td>true</td>
</tr>
<tr>
<td>((2, 5))</td>
<td>(x = 3)</td>
<td>false</td>
</tr>
<tr>
<td>([5, \infty))</td>
<td>(x = 6)</td>
<td>true</td>
</tr>
</tbody>
</table>

Therefore, we include \((-\infty, 2]\) and \([5, \infty)\) in the solution set. We do not need to check the endpoints. However, neither \( x = 2.5 \) works, since we are only seeking \( > 0 \) in the original equation. It is not true that \( 0 > 0 \). Therefore, the solution set is

\((-\infty, 2]\) \cup (5, \infty) \]

---

**Rational Inequalities**

Definition: A rational inequality is an inequality involving a rational expression.

To solve:

1. Rewrite the inequality so that we have a single fraction on one side and zero on the other.
2. Find values of \( x \) that will make the numerator zero. Find values of \( x \) that will make the denominator zero. These values will be the critical points.
3. Test a value from each interval to decide which intervals are included in the solution set.
\[ \frac{3}{x-6} \leq 2 \]

We begin by subtracting 2 from both sides and finding a common denominator:

\[
\frac{3}{x-6} - 2 \leq 0 \]

\[
\frac{3 - 2(x-6)}{x-6} \leq 0 \]

\[
\frac{2x + 12}{x-6} \leq 0 \]

\[
\frac{3 - 2x + 12}{x-6} \leq 0 \]

\[
\frac{-2x + 15}{x-6} \leq 0 \]

We find values that make the numerator and denominator equal to 0:

\[-2x + 15 = 0\]

\[x = \frac{15}{2}\]

\[x - 6 = 0\]

\[x = 6\]

Therefore, we have three intervals and create test points:

<table>
<thead>
<tr>
<th>Interval</th>
<th>Test Point</th>
<th>Is the inequality true at this point?</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-\infty, 6))</td>
<td>(x = 0)</td>
<td>true</td>
</tr>
<tr>
<td>((2, 7.5))</td>
<td>(x = 3)</td>
<td>false</td>
</tr>
<tr>
<td>((7.5, \infty))</td>
<td>(x = 8)</td>
<td>true</td>
</tr>
</tbody>
</table>

Now, we note that the inequality is true at \(x = 15/2\), but we must exclude \(x = 6\) since this makes the denominator 0. Therefore, the solution set is \((-\infty, 6) \cup (15/2, \infty)\)

### 95) Height of a Projectile

A projectile is fired straight up from ground level. After \(t\) seconds, its height above the ground is \(s\) feet, where

\[ s = -16t^2 + 220t \]

For what time period is the projectile at least 624 ft above the ground?

We begin with the inequality

\[ s \geq 624 \]

which means

\[ -16t^2 + 220t \geq 624 \]

We replace the inequality with \(a = -16\) and look for the critical points:

\[ -16t^2 + 220t - 624 = 0 \]

This is a quadratic, which we can solve by any of the available methods. We obtain the critical points

\[ t = 4, 9.75 \]

At this point we could investigate intervals and test points, but it is worthwhile to think about the physical situation. The projectile is at least 624 ft above the ground between 4 and 9.75 seconds (i.e. \([4, 9.75]\)). Why is this?
Absolute Value Equations and Inequalities, 1.8

- distance of a number from zero
- always equal to a positive number (represents a distance)

Properties of Absolute Value
1. \(|a| = -b \iff a = -b \text{ or } a = b \) (for \(b > 0\))
2. \(|a| = |b| \iff a = b \text{ or } a = -b\)

\#44) \(|8-3x| - 3 = -2\)

\(|8-3x| = 1\)

8-3x = 1  \hspace{1cm} 8-3x = -1

-3x = -7  \hspace{1cm} -3x = -9

x = \frac{7}{3}  \hspace{1cm} x = 3

Solution: (7/3, 3)

\#19) \(|2x-3| = |5x + 4|\)

4 Cases:
- a. 2x-3 = 5x + 4
- b. 2x-3 = -(5x + 4)
- c. -(2x-3) = 5x + 4 (Same as b)
- d. -(2x-3) = -(5x + 4) (Same as a)

So we only need to solve a) and b)

- a. 2x-3 = 5x + 4

-3x = 7

x = \frac{-7}{3}

- b. 2x-3 = -(5x + 4)

2x-3 = -5x-4

7x = -1

x = \frac{-1}{7}

\left[ \frac{-7}{3}, \frac{-1}{7} \right]

\#20) \(|x| = -7\)

Can an absolute value ever be negative? No

No solution

\emptyset
18) \( \frac{2x+3}{3x-4} - 1 \)

**NOTE:** \( x = \frac{4}{3} \) makes the denominator zero

\[
\begin{align*}
\frac{2x+3}{3x-4} &= 1 \\
\frac{2x+3}{3x-4} &= -1 \\
2x + 3 &= 3x - 4 \\
2x + 3 &\neq -3x + 4 \\
-x &= -7 \\
2x + 3 &= 3x + 4 \\
5x &= -1 \\
x &= \frac{1}{5}
\end{align*}
\]

Check solutions. Do they make the denominator zero? No.

Solution: \([\frac{4}{3}, 7]\)

---

**Inequalities with Abs. Value:**

**#5** \(|x| < 7\) means

![Number line for |x| < 7]

**#4** \(|x| > 7\) means

![Number line for |x| > 7]

**Properties of Absolute Value**

For any positive number \( b \):

3. \(|a| < b \) if and only if \(-b < a < b\)

4. \(|a| > b \) if and only if \(a < -b \) or \(a > b\)

---

**#28** \(|3x + 4| \leq 2\)

\[
\begin{align*}
-2 &\leq 3x + 4 < 2 \\
-6 &< 3x < 2 \\
-\frac{6}{3} &< x < -\frac{2}{3} \\
-2 &< x < \frac{2}{3}
\end{align*}
\]

Interval notation \((-2, \frac{2}{3})\)

---

**#52** \(|12 - 6x| + 3 > 9\)

\[
\begin{align*}
|12 - 6x| &> 6 \\
12 - 6x &\geq 6 \\
12 - 6x &\leq -6 \\
-6x &\geq -6 \\
-6x &\leq -18 \\
x &\leq 1 \\
x &\geq 3
\end{align*}
\]
Special Cases of Absolute Value Inequalities

Case 1: \(|a| > -b\) or \(|a| \geq -b\) where \(b > 0\)
   All real numbers
   \((-\infty, \infty)\)

Case 2: \(|a| < -b\) or \(|a| \leq -b\) where \(b > 0\)
   No solution
   \(\emptyset\)

Case 3: \(|a| < 0\)
   No Solution
   \(\emptyset\)

Case 4: \(|a| \geq 0\)
   All real numbers
   \((-\infty, \infty)\)

Case 5: \(|a| > 0\)
   All real numbers except value(s) that makes \(a = 0\)

Case 6: \(|a| \leq 0\)
   Only the value(s) that makes \(a = 0\) is a solution

\[\#56\] \(|12 - 9x| \geq -12\)
   All real numbers
   \((-\infty, \infty)\)

\[\#58\] \(|18 - 3x| < -13\)
   No solution
   \(\emptyset\)

\[\#64\] \(|3x + 2| \leq 0\)
   Only the value that makes \(3x + 2 = 0\) works
   \(3x + 2 = 0\)
   \(3x = -2\)
   \(x = \frac{-2}{3}\)

   The solution is \(\{\frac{-2}{3}\}\)

\[\#66\] \(|4x + 3| > 0\)
   All solutions work except \(4x + 3 = 0\)
   \(4x + 3 = 0\)
   \(4x = -3\)
   \(x = \frac{-3}{4}\)
Solution: \((-\infty, \frac{23}{5}) \cup (\frac{24}{5}, \infty)\)

#8.4) Write the statement as an absolute value inequality.

\(k\) is within .0002 unit of 7

\(|k-7| < .0002\)
Linear Inequalities / Interval Notation

Section 1.7

Solving Two-Part

When solving two-part linear inequalities (such as \(-4x + 5 > 13\)), you treat them like equalities (\(=\)) except when dividing or multiplying by a negative number. In that case, you have to reverse the inequality sign. Let’s take a look at solving an inequality for \(x\):

<table>
<thead>
<tr>
<th>Step:</th>
<th>Equality</th>
<th>Inequality</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subtract 5 from both sides</td>
<td>(-4x = 8)</td>
<td>(-4x &gt; 8)</td>
</tr>
<tr>
<td>Divide both sides by -4</td>
<td>(x = -2)</td>
<td>(x &lt; -2)</td>
</tr>
</tbody>
</table>

Note sign change since dividing by negative

Solving Three-Part

Solving three-part inequalities (like \(-9 \leq -6x +3 < 21\)) are the same as two-part, only you operate on all three parts when solving.

<table>
<thead>
<tr>
<th>Step:</th>
<th>(-9 \leq -6x +3 &lt; 21)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subtract 3 from all parts</td>
<td>(-12 \leq -6x &lt; 18)</td>
</tr>
<tr>
<td>Divide all three parts by -6</td>
<td>(2 \geq x &gt; -3)</td>
</tr>
<tr>
<td>Reorder</td>
<td>(-3 &lt; x \leq 2)</td>
</tr>
</tbody>
</table>

*This last step (not usually necessary) puts the inequality in the correct form, where the number on the left is the smallest, and the number on the right is the largest; make sure all inequality signs ‘point’ to the left. If you swap the numbers, you also swap and reverse the signs.
Interval Notation

After solving a linear inequality, you will get an expression like \( x < -2 \) or \( -3 < x \leq 2 \). To put this in interval notation, we use the following rules:

<table>
<thead>
<tr>
<th>Inequality Sign</th>
<th>Interval Sign</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt; or &gt;</td>
<td>( or )</td>
<td>Less/Greater than (but not equal to)</td>
</tr>
<tr>
<td>( \leq ) or ( \geq )</td>
<td>[ or ]</td>
<td>Less/Greater than or equal to</td>
</tr>
</tbody>
</table>

*An easy way to remember this is if you have a straight line under the inequality, you use the straight line bracket.*

When writing in interval notation, you always start with the smallest number on the left and the largest number on the right.

If you have \( x < \# \) or \( x \leq \# \), then \(-\infty\) will be used on the left-hand side of the interval. Likewise if \( x > \# \) or \( x \geq \# \), then \( \infty \) will be used on the right-hand side of the interval. **Always use ( or ) for -\( \infty \) or \( \infty \).**

Here are some examples:

<table>
<thead>
<tr>
<th>Inequality Notation</th>
<th>Interval Notation</th>
<th>Number Line</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x &lt; -2 )</td>
<td>((-\infty,-2))</td>
<td></td>
</tr>
<tr>
<td>( x \geq -5 )</td>
<td>([-5, \infty))</td>
<td></td>
</tr>
<tr>
<td>( 4 &lt; x \leq 13 )</td>
<td>((4, 13])</td>
<td></td>
</tr>
</tbody>
</table>

In some cases, you will have a disjoint interval and so you will use the union symbol U to combine the different parts of the interval:

<table>
<thead>
<tr>
<th>Inequality Notation</th>
<th>Interval Notation</th>
<th>Number Line</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x &lt; -2 ) or ( x &gt; -1 )</td>
<td>((-\infty,-2)U(-1,\infty))</td>
<td></td>
</tr>
<tr>
<td>( x &lt; -8 ) or ( x \geq 5 )</td>
<td>((-\infty,-8)U[5, \infty))</td>
<td></td>
</tr>
</tbody>
</table>
Quadratic Inequalities

When we solve equations, we are looking for a specific value of a variable that makes a statement true. The same is true for inequalities. We want to know what values of ‘x’ can be plugged into a statement to make it true. For equations, it is a set of values. For inequalities, it may be an interval or union of intervals of values.

So how do we solve quadratic and polynomial inequalities? The same way we solve all inequalities: we find the critical points that will break up the number line into intervals. Then we use test points in those intervals to see if they work.

Example: Solve \( x^2 - 4 < 0 \).

Inequalities can only change sign at critical points, which is where the expressions on both sides of the inequality symbol can be undefined, OR when the entire inequality is set equal to zero and then solved. We won’t have to worry about division by zero or radicals when solving polynomial inequalities, so we are just looking for zeros now. So here we solve \( x^2 - 4 = 0 \), and we factor the difference of squares to get \( (x + 2)(x - 2) = 0 \), so we see -2 and 2 are the critical points. Since there are two critical points, that breaks up the number line into three intervals. Let’s find test points in those intervals and see if the inequality holds.

<table>
<thead>
<tr>
<th>Interval</th>
<th>Test point</th>
<th>Sign</th>
<th>True/False</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-\infty, -2))</td>
<td>-3</td>
<td>+ (since 9 – 4 is positive)</td>
<td>False (we want less than zero)</td>
</tr>
<tr>
<td>((-2, 2))</td>
<td>0</td>
<td>- (since 0 – 4 is negative)</td>
<td>True</td>
</tr>
<tr>
<td>((2, \infty))</td>
<td>3</td>
<td>+ (since 9 – 4 is positive)</td>
<td>False</td>
</tr>
</tbody>
</table>

How did we know the intervals were all using parentheses? Because of the < sign. If it were \( \leq \) or \( \geq \) we could allow those critical points. But if we put in -2 or 2 in this example, we get 0 < 0, which is a false statement. So now we can just take the interval that works, which will be our final answer: \((-2, 2)\).

Example: Solve \( 2x^3 - x \geq 0 \).

We don’t have to worry about critical points where the inequality is undefined, so we can solve the corresponding equation: \( 2x^3 - x = 0 \)
\[ x(2x^2 - 1) = 0 \]
\[ x = 0 \quad 2x^2 - 1 = 0 \]
\[ x^2 = \frac{1}{2} \quad \Rightarrow \quad x = \pm \frac{1}{\sqrt{2}} \quad \Rightarrow \quad x = \pm \frac{\sqrt{2}}{2} \]

So we have three critical points. This will break up the number line into four intervals.

<table>
<thead>
<tr>
<th>Interval</th>
<th>Test point</th>
<th>Sign</th>
<th>True/False</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-\infty, -\frac{\sqrt{2}}{2}))</td>
<td>-1</td>
<td>-2 + 1, negative</td>
<td>False</td>
</tr>
<tr>
<td>([-\frac{\sqrt{2}}{2}, 0])</td>
<td>-0.5</td>
<td>(-\frac{1}{4} + \frac{1}{2} \geq 0)</td>
<td>True</td>
</tr>
<tr>
<td>([0, \frac{\sqrt{2}}{2}])</td>
<td>(\frac{1}{2})</td>
<td>(-\frac{1}{4} - \frac{1}{2}), negative</td>
<td>False</td>
</tr>
<tr>
<td>([-\frac{\sqrt{2}}{2}, \infty))</td>
<td>1</td>
<td>2 - 1, positive</td>
<td>True</td>
</tr>
</tbody>
</table>

Notice that we used brackets everywhere (except at infinity, which is always use parentheses) for our intervals. That’s because we’re dealing with a \(\geq\), not just a >. So now our solution is just a union of the intervals that work. Our answer is \([−\frac{\sqrt{2}}{2}, 0] \cup \left[\frac{\sqrt{2}}{2}, \infty\right]\).

Example: Solve \(x^2 + 7x + 2 \leq 0\).
Since we have no points where our inequality is undefined, our only critical points will come from zeros of the corresponding equation \(x^2 + 7x + 2 = 0\). So we can use the quadratic formula to solve for \(x\):

\[
x = \frac{-7 \pm \sqrt{7^2 - 4(1)(2)}}{2(1)}
\]
\[= \frac{-7 \pm \sqrt{49 - 8}}{2}
\]
\[= \frac{-7 \pm \sqrt{41}}{2}
\]

So our critical points are \(-\frac{7 - \sqrt{41}}{2}\) and \(-\frac{7 + \sqrt{41}}{2}\). If we put this on a calculator we can get approximation (so we know where to test), \(-\frac{7 - \sqrt{41}}{2} \approx -6.7\) and \(-\frac{7 + \sqrt{41}}{2} \approx -0.30\). So if we
need a number less than -6.7, we can choose -7. Between -6.7 and -.3, we can choose -1. And greater than -.3 we can just go with 0. So now we test:

\[ x^2 + 7x + 2 \leq 0 \]

<table>
<thead>
<tr>
<th>Interval</th>
<th>Test point</th>
<th>Sign</th>
<th>True/False</th>
</tr>
</thead>
<tbody>
<tr>
<td>( -\infty, \frac{-7 - \sqrt{41}}{2} )</td>
<td>-7</td>
<td>49 - 49 + 2 is positive, which is not less than 0</td>
<td>False</td>
</tr>
<tr>
<td>( \frac{-7 - \sqrt{41}}{2}, \frac{-7 + \sqrt{41}}{2} )</td>
<td>-1</td>
<td>1 - 7 + 2 is negative, which is less than 0</td>
<td>True</td>
</tr>
<tr>
<td>( \frac{-7 + \sqrt{41}}{2}, \infty )</td>
<td>0</td>
<td>0 + 0 + 2 is positive, which is not less than 0</td>
<td>False</td>
</tr>
</tbody>
</table>

Notice again that the less than or equal to sign allows us to use brackets when doing interval notation. So we take the union of the intervals that work (but it’s just one interval), so our answer is \( \left[ -\frac{7 - \sqrt{41}}{2}, -\frac{7 + \sqrt{41}}{2} \right] \).

Example: Solve \( x^3 - 8 > 0 \).

There are no points where the inequality is undefined since polynomials are defined for all real numbers. So we can solve the corresponding equation \( x^3 - 8 = 0 \). We will factor using the difference of cubes formula: \( a^3 - b^3 = (a-b)(a^2 + ab + b^3) \). So we have

\[ x^3 - 8 = (x-2)(x^2 + 2x + 4) = 0 \]

We have 2 as a critical point. The quadratic is usually never factorable, so we can use the quadratic formula to find the roots:

\[ x = \frac{-2 \pm \sqrt{4 - 4(1)(4)}}{2} = \frac{-2 \pm \sqrt{-12}}{2} \]

These are non-real roots, so we can’t put them on the real number line. So what do we do? We forget them. So our only critical point is 2, so we only have two intervals to test:

<table>
<thead>
<tr>
<th>Interval</th>
<th>Test point</th>
<th>Sign</th>
<th>True/False</th>
</tr>
</thead>
<tbody>
<tr>
<td>( -\infty, 2 )</td>
<td>0</td>
<td>(-) which is not greater than zero</td>
<td>False</td>
</tr>
<tr>
<td>( 2, \infty )</td>
<td>3</td>
<td>(+) which is greater than zero</td>
<td>True</td>
</tr>
</tbody>
</table>

We used parentheses in our intervals because we cannot allow 2 to be in the solution. We would end up with \( 0 > 0 \), which is a false statement. So we take the interval that works as our answer: \( (2, \infty) \).
Rational Inequalities

Section 1.7

When solving rational inequalities (x appears in the denominator), such as the one below:

\[ \frac{10}{3 + 2x} \leq 5 \]

it is important to note that you never multiply both sides by the denominator since it could be positive or negative depending on the value of x, so you would not know whether to reverse the inequality sign or not.

To get around this, move everything to one side and find a common denominator to get a single rational expression less/greater than 0.

<table>
<thead>
<tr>
<th>Step:</th>
<th>( \frac{10}{3 + 2x} \leq 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subtract 5 from both sides</td>
<td>( \frac{10}{3 + 2x} - 5 \leq 0 )</td>
</tr>
<tr>
<td>Multiply the -5 by ( \frac{3+2x}{3+2x} ) to get common denominator</td>
<td>( \frac{10 - 5(3 + 2x)}{3 + 2x} \leq 0 )</td>
</tr>
<tr>
<td>Combine into one fraction</td>
<td>( \frac{10 - 15 - 10x}{3 + 2x} \leq 0 )</td>
</tr>
<tr>
<td>Distribute; careful to distribute the negative sign</td>
<td>( \frac{-5 - 10x}{3 + 2x} \leq 0 )</td>
</tr>
<tr>
<td>Simplify</td>
<td></td>
</tr>
</tbody>
</table>
Now that we have the inequality in this form, we still need to solve it. We first find the **zeros of the numerator and denominator**, then **test each interval** in between them to determine the solution.

<table>
<thead>
<tr>
<th>Numerator:</th>
<th>Denominator:</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-5 - 10x = 0)</td>
<td>(3 + 2x = 0)</td>
</tr>
<tr>
<td>(-10x = 5)</td>
<td>(2x = -3)</td>
</tr>
<tr>
<td>(x = -\frac{1}{2})</td>
<td>(x = -\frac{3}{2})</td>
</tr>
</tbody>
</table>

So the critical points are \(-1/2\) and \(-3/2\). The number line is split at those points:

Our three intervals are then \((-\infty, -\frac{3}{2}) (-\frac{3}{2}, -\frac{1}{2}) [-\frac{1}{2}, \infty)\). We use square brackets \([\) and \(]\) for the ends of the intervals since our inequality has a \(\leq\) sign. However, since \(-3/2\) is the zero of the denominator, we cannot include it in our solution, so we use rounded parentheses \((\) and \())\).

Now we plug **any** number within each interval A, B, and C into our inequality and see which intervals gives us a true expression. It is best to use the simplified version of the inequality \((\leq 0)\) to avoid mistakes, such as when you are trying to determine if, for example, \(-\frac{12}{7} \leq -\frac{17}{10}\) is a true statement.
Interval: | Point in Interval | Plug into inequality | Is inequality true? |
---|---|---|---|
A: $\left( -\infty, -\frac{3}{2} \right)$ | -2 | $\frac{-5 - 10(-2)}{3 + 2(-2)} \leq 0 \rightarrow -15 \leq 0$ | Yes |
B: $\left( -\frac{3}{2}, -\frac{1}{2} \right]$ | -1 | $\frac{-5 - 10(-1)}{3 + 2(-1)} \leq 0 \rightarrow 5 \leq 0$ | No |
C: $\left[ -\frac{1}{2}, \infty \right)$ | 0 | $\frac{-5 - 10(0)}{3 + 2(0)} \leq 0 \rightarrow \frac{-5}{3} \leq 0$ | Yes |

*We could have chosen -10, -1.25, and 13 as our three points but -2, -1, and 0 are just easier to plug in.

Our answer is the union of all the intervals that work (a Yes in the last column), so $\left( -\infty, -\frac{3}{2} \right) \cup \left[ -\frac{1}{2}, \infty \right)$ is our solution.

Here is another example when you have $x$ in two of the denominators. Like before, we need to move everything to one side (rather than multiply through by the LCD).

<table>
<thead>
<tr>
<th>Step:</th>
<th>$\frac{4}{2-x} \geq \frac{3}{x+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subtract the $\frac{3}{x+1}$ from both sides</td>
<td>$\frac{4}{2-x} - \frac{3}{x+1} \geq 0$</td>
</tr>
<tr>
<td>Multiply $\frac{4}{2-x}$ by $\frac{x+1}{x+1}$ and $\frac{3}{x+1}$ by $\frac{2-x}{2-x}$ to get common denominator</td>
<td>$\frac{4(x+1)}{2-x(x+1)} - \frac{3(2-x)}{x+1(2-x)} \geq 0$</td>
</tr>
<tr>
<td>Combine into one fraction</td>
<td>$\frac{4(x+1) - 3(2-x)}{(2-x)(x+1)} \geq 0$</td>
</tr>
<tr>
<td>Distribute; careful to distribute the negative sign</td>
<td>$\frac{4x + 4 - 6 + 3x}{(2-x)(x+1)} \geq 0$</td>
</tr>
<tr>
<td>Simplify</td>
<td>$\frac{7x - 2}{(2-x)(x+1)} \geq 0$</td>
</tr>
</tbody>
</table>

Now we find the zeros of the numerator and denominator.
Numerator: \[7x - 2 = 0\]  
Denominator: \[2 - x = 0 \quad \text{and} \quad x + 1 = 0\]

\[
x = \frac{2}{7} \quad \text{and} \quad x = 2 \quad \text{and} \quad x = -1
\]

So our critical points are \( \frac{2}{7}, 2 \) and \(-1\). Here is the number line:

![Number line diagram]

Notice that now we have 4 intervals because we have 3 critical points. They are \((-\infty, -1) \quad (-1, \frac{2}{7}) \quad \left[\frac{2}{7}, 2\right) \quad (2, \infty)\). Recall -1 and 2 are zeros of the denominator, so they are not included and use rounded parenthesis. The \(\frac{2}{7}\) (zero of numerator) is included since our original equation has a \(\geq\) sign, so we use a square bracket around that point.

<table>
<thead>
<tr>
<th>Interval:</th>
<th>Point in Interval</th>
<th>Plug into inequality</th>
<th>Is inequality true?</th>
</tr>
</thead>
<tbody>
<tr>
<td>A: ((-\infty, -1))</td>
<td>-2</td>
<td>[\frac{7(-2) - 2}{(2 - (-2))((-2) + 1)} \geq 0]</td>
<td>Yes</td>
</tr>
<tr>
<td>B: ((-1, \frac{2}{7}))</td>
<td>0</td>
<td>[\frac{7(0) - 2}{(2 - (0))((0) + 1)} \geq 0]</td>
<td>No</td>
</tr>
<tr>
<td>C: (\left[\frac{2}{7}, 2\right))</td>
<td>1</td>
<td>[\frac{7(1) - 2}{(2 - (1))((1) + 1)} \geq 0]</td>
<td>Yes</td>
</tr>
<tr>
<td>D: ((2, \infty))</td>
<td>3</td>
<td>[\frac{7(3) - 2}{(2 - (3))((3) + 1)} \geq 0]</td>
<td>No</td>
</tr>
</tbody>
</table>

So our answer is the interval \((-\infty, -1) \cup \left[\frac{2}{7}, 2\right)\).
Absolute Value

***Please read at least the tricks at the end of the handout***

Absolute value means “distance from zero.” So when we take the absolute value of a number, we are asking what it’s distance from zero on the real number line is. In that way absolute value is always positive. \(|3| = 3\), and \(|-3| = 3\), since the distance from zero to 3 is 3, and the distance from zero to -3 is also 3. Absolute value is distance, and distance is always positive.

So when we are solving an absolute value equation like \(|x| = 2\), we say the distance between 0 and x is 2. The solution will be x = -2 or x = 2. So there will be two corresponding equations when you solve an absolute value equation. We summarize this in the following manner.

- To solve \(|a| = b\), set \(a = b\) and \(a = -b\).

Example: Solve \(|8 - 3x| = -2\).
Like every equation, our goal is to isolate the x. The first thing we have to do now is to isolate the absolute value so we can use our method. So let’s add 3 to both sides to get \(|8 - 3x| = 1\).

Now we have \(8 - 3x = 1\) and \(8 - 3x = -1\). So when we solve these two equations we have

\[
\begin{align*}
3x &= 7 \\
3x &= 9
\end{align*}
\]

\[
\Rightarrow x = \frac{7}{3} \quad \text{and} \quad x = 3
\]

So our solution set is \(\{\frac{7}{3}, 3\}\).

Now let’s consider absolute value inequalities. Consider the inequality \(|x| < 7\). We know absolute value means distance, and x can represent any number. So read that statement aloud: “The absolute value of x is less than 7.” The translation is that we are looking for numbers whose distance from 0 is less than 7. Well, that includes all \(x < 7\), but we include all \(x > -7\) as well. So the solution to the inequality \(|x| < 7\) is \(-7 < x < 7\), or \((-7, 7)\). Notice the use of parentheses because -7 and 7 are not included.

- To solve \(|a| < b\), set \(a < b\) and \(a > -b\) and solve those inequalities.

Example: Solve \(|x| > 7\).
We have \( x > 7 \), and then we do the “flip-op” \( x < -7 \). Since -7 and 7 are not included, we can write the solution in interval notation:\(( -\infty, -7) \cup (7, \infty)\).

Example: \( |2-6x|+3 \geq 9 \).

We isolate the absolute value (so we can isolate the \( x \)), so we can subtract 6 from both sides to get \( |2-6x| \geq 6 \). So we solve the inequality written like it is without the absolute value bars:
\[
12-6x \geq 6, \quad 12-6x \leq -6.
\]

So let’s solve these inequalities.
\[
12-6x \geq 6 \quad \Rightarrow \quad -6x \geq -6 \quad \Rightarrow \quad x \leq 1
\]

\[
-6x \leq -18 \quad \Rightarrow \quad x \geq 3
\]

Remember when you are solving inequalities, if you divide (or multiply) by a negative number, you MUST flip the sign. So any number less than or equal to 1, or larger than or equal to 3 will work. So using interval notation we write the answer as \(( -\infty, 1] \cup [3, \infty) \). Notice the brackets in the answer because we can allow 1 and 3 into the equality, because \( 9 \geq 9 \).

There are a lot of tricks that appear with absolute value.

Example: Solve \( |x| = -7 \).

You might say \( x = -7 \) and \( x = 7 \), but if you plug those answers back into the equation, they don’t work. Remember, absolute value is always positive. For \( x \) any real number we can say \( |x| \geq 0 \).

There is no solution to this problem (the answer is the empty set).

Example: Solve \( |2-9x| \geq -12 \).

If you solve this normally, you write the inequality and solve: \( 12-9x \geq -12 \). \( 9x \leq 24 \), \( x \leq \frac{24}{9} \).

Then you flip the inequality sign, take the opposite and solve again. So we have
\[
12-9x \leq 12 \quad \Rightarrow \quad 9x \geq 24 \quad \Rightarrow \quad x \geq \frac{24}{9}.
\]

So we have \( x \) can be any number less than or equal to \( \frac{24}{9} \), or \( x \) can be any number greater than or equal to \( \frac{24}{9} \). That’s all the numbers! How did this happen? The answer is in the statement of the question. When is the absolute value of something going to be bigger than a negative number? Always. So whenever we have something like this, we can say the solution is all real numbers, or \(( -\infty, \infty) \).

Example: Solve \( |8-3x| < -13 \).

If you try to solve this inequality, you will be trying to find where a positive number will be less than a negative number. That’s never going to happen. So the answer is the empty set: \( \emptyset \).
Textbook Section 2.1
Rectangular Coordinates and Graphs

Objectives

- The student will be able to find ordered pairs and graph equations
- The student will be able to use distance and midpoint formulas to solve problems
- The student will be able to decide whether points are collinear or vertices of a right triangle
- The student will be able to solve application problems

Key Concepts

Distance Formula:

distance between points P and R

Point #1: \( P(x_1, y_1) \)
Point #2: \( Q(x_2, y_2) \)

\[ d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]

Midpoint Formula:

For a line segment with endpoints \((x_1, y_1)\) and \((x_2, y_2)\)
the midpoint has coordinates \(\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)\).

\( x\)-intercept = Point where the graph touches the \(x\)-axis
To find: set \(y = 0\) and solve for \(x\)

\( y\)-intercept = Point where the graph touches the \(y\)-axis
To find: set \(x = 0\) and solve for \(y\)

Textbook Section 2.2
Circles

Objectives

- The student will be able to find center-radius form of a circle
- The student will be able to convert an equation in general form to center-radius form
- The student will be able to find equation of a circle using the graph
- The student will be able to find the center, radius, and graph of a circle
- The student will be able to solve application problems

Key Concepts

Definition:
A circle is the set of all points in a plane that lie a given distance (radius) from a given point (center).

Center-Radius Form of the Equation of a Circle

A circle with center \((h, k)\) and radius \(r\) has the equation:

\[ (x - h)^2 + (y - k)^2 = r^2 \]

\textit{NOTE:} A circle with center \((0,0)\) and radius \(r\) has the equation \(x^2 + y^2 = r^2\).

General Form of the Equation of a Circle
For some real numbers $c$, $d$, and $e$, the equation

$$x^2 + y^2 + cx + dy + e = 0$$

can have a graph that is a circle, a point, or is nonexistent.

**Circle:** radius = positive number

**Point:** radius = 0

**Non-existent:** radius = negative number

To convert to Center-Radius Form,

- Complete the square for both $x$ and $y$.

---

**Textbook Section 2.3**

**Functions**

**Objectives**

- The student will be able to decide if a relation is a function
- The student will be able to find the domain and range of a function
- The student will be able to use function notation
- The student will be able to evaluate a function
- The student will be able to find intervals of the domain where a function is increasing, decreasing, or constant

**Key Concepts**

**Definition:**

A relation is a set of ordered pairs.

- A relationship showing how one quantity depends on another.

**Definition:**

A *function* is a relation in which, for each distinct value of the first component ($x$), there is exactly one value of the second component ($y$).

**Vertical Line Test:**

If each vertical line intersects a graph in at most one point, then the graph is that of a function.

- $x$ is called: input, independent variable
- $y$ is called: output, dependent variable

**Domain** = set of all $x$-values that produce real number $y$-values

**Range** = set of all $y$-values

**Finding Domain from a Formula:**

- Look for restrictions. (What $x$ makes $y$ undefined?)
- Common restrictions:
  - denominator of a rational function cannot = 0
  - argument of a square root must be non-negative

We say "$y$ is a function of $x" to emphasize that $y$ depends on $x$.

**Function Notation:**

When function $f$ is applied to an input $x$,

we write $f(x)$ to represent the resulting output $y$.

Notation: Write $y = f(x)$. Read "$f$ of $x".

**Increasing, Decreasing, and Constant Functions**

Suppose that a function $f$ is defined over an interval, $I$.

If $x_1$ and $x_2$ are in $I$,

- $f$ increases on $I$ if, whenever $x_1 < x_2$, $f(x_1) < f(x_2)$;
- $f$ decreases on $I$ if, whenever $x_1 < x_2$, $f(x_1) > f(x_2)$;
- $f$ is constant on $I$ if, for every $x_1$ and $x_2$, $f(x_1) = f(x_2)$.
Textbook Section 2.4
Linear Function

Objectives

- The student will be able to graph a linear function
- The student will be able to graph a horizontal or vertical line, given an equation
- The student will be able to find slope and graph given an equation
- The student will be able to graph a line given point and slope
- The student will be able to find and interpret average rate of change
- The student will be able to solve applications

Key Concepts

Linear function:

\[ f(x) = ax + b. \]

Slope of a line through the points \((x_1, y_1)\) and \((x_2, y_2)\) is

\[ m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}, \]

where \(\Delta x \neq 0\)

Possible Results:

- Positive Slope: rising line
- Negative Slope: falling line
- Slope is 0: horizontal line
- Undefined Slope: vertical line

Standard Form:

\[ Ax + By = C \]

where A, B, and C are integers and \(A > 0\)

where the greatest common factor between A, B, and C is 1.

Average Rate of Change:

The slope gives the average rate of change in \(y\) per unit of change in \(x\).

Handouts and Applets

Some of the files you are about to view/download are PDF files. If you do not have Adobe Acrobat installed on your system, you can download the free Adobe Acrobat Reader at http://www.adobe.com/products/acrobat/alternate.html

- Equations of Lines Handout
- Graphing a line using y-intercept and slope applet: http://www.itcc.online.net/green/java/BasicAlgebra/LineGraph/LineGraph.htm
- Completing the Square Handout

Online Homework and Quiz Assignments

After you have reviewed the Key Concepts and Handouts, log into MyLabsPlus and begin your homework and quiz for this week, go to www.ucf.mylabsplus.com and begin working on your assignments.

Reminders

You have assignments due during test week!
Rectangular Coordinates and Graphs, 2.1

Definition: An ordered pair consists of two components. We usually write them as \((x, y)\). The order of the components is important!

This is called the xy-plane, the rectangular coordinate system, or the Cartesian coordinate system, after René Descartes, a French philosopher and mathematician.

The plane, which is infinite, is useful because it allows us to visualize ordered pairs. For obvious reasons, we will only ever see portions of the plane. Here, for instance, we view \(x\) and \(y\) from \(-10\) to \(10\).

The plane is split into four quadrants, here marked I–IV, and has two axes. The \(x\)-axis is horizontal and runs from \(-\infty\) on the left to \(\infty\) (usually just written as \(\infty\)) on the right. The \(y\)-axis is vertical and runs from \(-\infty\) down to \(\infty\) up. The origin is where these two axes intersect, the point \((0, 0)\).

The objects in red will not usually be marked and are here only for convenience.

Exercise: #8)

Give three ordered pairs from the table and graph them.

\[
\begin{array}{c|c}
 x & y \\
\hline
 3 & 3 \\
 4 & 8 \\
 0 & -6 \\
 -5 & -21 \\
 8 & 16 \\
\end{array}
\]

Distance Formula: Given two points in the plane

\[p_1 = (x_1, y_1)
\]
\[p_2 = (x_2, y_2)
\]

We might want to find the distance between \(p_1\) and \(p_2\). Using the Pythagorean theorem, we can prove that

\[
d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}
\]

Exercise: #12a)

Find the distance between \(P(4, 3)\), \(Q(2, -5)\).

Solution:

\[
d(P, Q) = \sqrt{(4 - 2)^2 + (3 - (-5))^2}
\]
\[
d(P, Q) = \sqrt{6^2 + (-8)^2}
\]
\[
d(P, Q) = \sqrt{36 + 64}
\]
\[
d(P, Q) = \sqrt{100}
\]
\[
d(P, Q) = 10
\]

So the distance between the points is 10 units.
If we are given three points, \(P\), \(Q\), and \(R\), these will usually form a triangle unless they are collinear, which we will define later. We might like to know if they form a right triangle. Instead of worrying with trigonometry, we know that if \(\triangle PQR\) is a right triangle, then the Pythagorean theorem must be true:

\[
(PQ)^2 + (QR)^2 = (PR)^2
\]

where \(PQ\), for instance, means the distance between \(P\) and \(Q\). \(\textbf{NOTE:} PR\) is the longest side.

---

**Exercise: #20**

Determine whether the three points are the vertices of a right triangle:

\(P(2,-8), Q(0,4), R(-4,7)\)

Solution: We begin by calculating the length of each side:

\[
PQ = \sqrt{(0 - (-2))^2 + (-4 - (-8))^2} \\
PQ = \sqrt{4^2 + 4^2} \\
PQ = \sqrt{16 + 16} \\
PQ = \sqrt{32}
\]

\[
PR = \sqrt{(-4 - (-2))^2 + (-7 - (-8))^2} \\
PR = \sqrt{(-2)^2 + 1^2} \\
PR = \sqrt{4 + 1} \\
PR = \sqrt{5}
\]

\[
QR = \sqrt{(-4 - 0)^2 + (-7 - (-4))^2} \\
QR = \sqrt{(-4)^2 + 3^2} \\
QR = \sqrt{16 + 9} \\
QR = \sqrt{25}
\]

The longest side is \(QR\). Note that it doesn’t matter which point we labeled what – since we always identify the longest side, the labels do not matter as long as they are consistent.

Is it true that \((PR)^2 + (PQ)^2 = (QR)^2\)? We calculate:

\[
(\sqrt{5})^2 + (\sqrt{32})^2 = (\sqrt{25})^2
\]

\[
5 + 32 = 25 \\
25 = 25
\]

This is true, so the points form a right triangle.

---

We can also ask if three points \(P\), \(Q\), and \(R\) are collinear – in other words, do they all lie on the same line? Again, we can use the distance formula. If the points \(P\), \(Q\), and \(R\) are collinear, then it must be true that

\[
PQ + QR = PR
\]

where \(PR\) is the longest length.

---

**Exercise: #25**

Determine whether the three points are collinear:

\(P(0,7), Q(3,5), R(2,15)\)

Solution: We begin by calculating the distances:

\[
PQ = \sqrt{(-3 - 0)^2 + (5 - (-7))^2} \\
PQ = \sqrt{(-3)^2 + (12)^2} \\
PQ = \sqrt{9 + 144} \\
PQ = \sqrt{153}
\]

---
\[ PR = \sqrt{(2 - 0)^2 + (-15 - (-7))^2} \]
\[ PR = \sqrt{2^2 + (-8)^2} \]
\[ PR = \sqrt{4 + 64} \]
\[ PR = \sqrt{68} \]

\[ QR = \sqrt{(2 - (-3))^2 + (-15 - 5)^2} \]
\[ QR = \sqrt{5^2 + (-20)^2} \]
\[ QR = \sqrt{425} \]

The longest side is QR. Is it true that \( PR + PQ = QR \)? We calculate:

\[ \sqrt{68} + \sqrt{133} = \sqrt{425} \]
\[ \sqrt{5^2 + 17^2} + \sqrt{9^2 + 17^2} = \sqrt{25 + 17} \]
\[ 2\sqrt{17} + 3\sqrt{17} = 5\sqrt{17} \]
\[ 5\sqrt{17} = 5\sqrt{17} \]

This is true, so the points are collinear.

Given two points
\[ P_1 = (x_1, y_1) \]
\[ P_2 = (x_2, y_2) \]
we might want to find the point halfway between them, the midpoint. We therefore have the midpoint formula:
\[ M = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \]

Exercise: #12b)

Find the coordinates of the midpoint of the segment PQ where
\[ P(-4, 3), Q(2, 5) \]

Solution:
\[ M = \left( \frac{-4+2}{2}, \frac{3+5}{2} \right) \]
\[ M = \left( \frac{-2}{2}, \frac{8}{2} \right) \]
\[ M = (-1, 4) \]

Therefore, \( M \) is halfway between \( P \) and \( Q \).

Exercise: #32)

A line segment has two endpoints and a midpoint. Here, we give you one endpoint and the midpoint. Find the other endpoint:

Midpoint: (-7, 6)  Endpoint: (-9, 9)

Solution: Call the missing endpoint
\[ P = (x_1, y_1) \]
Then we know that the midpoint must be
\[ M = \left( \frac{x_1 + (-9)}{2}, \frac{y_1 + 9}{2} \right) \]

However, the problem also tells us that the midpoint is (-7, 6). Therefore,
\[ (-7, 6) = \left( \frac{x_1 + (-9)}{2}, \frac{y_1 + 9}{2} \right) \]

This gives us two equations:
\[ -7 = \frac{x_1 + (-9)}{2} \quad \frac{6}{2} = \frac{y_1 + 9}{2} \]
\[-14 = x_1 + (-9) \quad 12 = y_1 + 9 \]
\[ x_1 = -5 \quad y_1 = 3 \]

Therefore, the second endpoint is (-5, 3)
Exercise: #38)

The graph shows an idealized linear relationship for the average monthly payment to needy families in the Temporary Assistance for Needy Families (TANF) program. Based on this information, what was the average payment to families in 2002?

Solution. We know the average monthly payment in 2000 and 2004. For instance, the average monthly payment in 2000 is $387. We can represent this as the point (2000, 387). We would like to find the monthly payment in 2002, which, conveniently, is in the middle of these two years. Therefore, we can use the midpoint formula on the two points (2000, 387), (2004, 500).

We calculate:

$$M = \left( \frac{2000 + 2004}{2}, \frac{387 + 500}{2} \right)$$

So the midpoint is (2002, 446.5). This means that the average payment to families in 2002 was $446.50.

Graphing an Equation by Point Plotting

We recall that one of the major uses of the xy-plane is to make it easier to understand the relationship between x and y. Graphing a finite number of points is simple. Often, though, we will be given an equation rather than a list of points.

To graph an equation, we first look for two special types of points. An x-intercept occurs whenever the graph touches the x-axis. To find the x-intercept, set y = 0 and solve for x. A y-intercept occurs whenever the graph touches the y-axis. To find the y-intercept, set x = 0 and solve for y.

In general:

Step 1: Find the intercepts.
Step 2: Find additional ordered pairs.
Step 3: Plot all ordered pairs.
Step 4: Connect with a smooth line or curve.

Exercise: #50)

For the following equation, give a table with at least three ordered pairs that are solutions, and graph the equation

$$y = \sqrt{x} - 3$$

Solution. To find the x-intercept(s), we set y = 0 and solve for x:

$$0 = \sqrt{x} - 3$$
$$3 = \sqrt{x}$$
$$3^2 = \left( \sqrt{x} \right)^2$$
$$9 = x$$

Therefore, we have the point (9, 0) on the graph. To find the y-intercept(s), we set x = 0 and find y:

$$y = \sqrt{0} - 3 = 0 - 3 = -3$$

so the point (0, -3) is on the graph. We try an additional point, letting x = 4:

$$y = \sqrt{4} - 3 = 2 - 3 = -1$$

so the point (4, -1) is also on the graph. We did not have to choose x = 4. Any other x would do, but since we are taking the square root of x, we chose a convenient x.

Therefore, we have the following table. We plot the points and connect them with a curved line.
Why does the graph stop at \( x = 0 \)? Why this particular curved line? This method only gives a rough sketch of the graph and additional knowledge is necessary if the graph needs to be more than an approximation.

Exercise: #4)

For the following equation, give a table with at least three ordered pairs that are solutions, and graph the equation

\[ 6y = -6x + 18 \]

Solution: In this case, it is convenient to first solve for \( y \):

\[ y = \frac{-6x + 18}{6} \]
\[ y = -x + 3 \]

We can then more easily plug in points and obtain our graph.
Circles, 2.2

Definition
A circle is the set of all points in a plane that lie a given distance (radius) from a given point (center).

There are several ways to describe a circle mathematically. The most immediately useful is the center-radius form:

\[(x - h)^2 + (y - k)^2 = r^2\]

which represents a circle with center \((h, k)\) and radius \(r\). In particular, if the circle is centered at the origin \((0,0)\) with radius \(r\), it has the equation

\[x^2 + y^2 = r^2\]

Exercise: #6
Find the center-radius form of the equation a circle with center \((4,3)\) and radius 5, then graph it.

Solution:
Since the center is \((h, k) = (4,3)\) and \(r = 5\), the center-radius form is:

\[(x-4)^2 + (y-3)^2 = 5^2\]
\[(x-4)^2 + (y-3)^2 = 25\]

We can, of course, multiply out the center-radius form, in which case we obtain the general form:

\[x^2 + y^2 + cx + dy + e = 0\]

where \(c, d,\) and \(e\) are some real numbers. Of course, if we start with a circle, it will always be a circle. If we're given a general form, however, it doesn't necessarily represent an actual circle; we have to convert general form into center-radius form to know. Once we have done that, there are three possibilities. The general form represents:

- a circle: if \(r^2 = \text{positive number}\)
- a point: if \(r^2 = 0\)
- nothing: if \(r^2 = \text{negative number}\)

To convert from general form into center-radius form:

1. Divide by the coefficient of \(x^2\). If the coefficients of both \(x^2\) and \(y^2\) do not become 1, the equation does not represent a circle and cannot be converted to center-radius form.
2. Complete the square in both \(x\) and \(y\).

Exercise: #19
Determine whether the equation has a circle as its graph. If it does, give the center and radius. If not, describe the graph.

\[x^2 + y^2 + 6x + 8y + 9 = 0\]

Solution:
Since the coefficient of \(x^2\) is 1, we do not have to divide by anything. For convenience, we move the constant term to the other side:

\[x^2 + 6x + y^2 + 8y = -9\]

We complete the square in both \(x\) and \(y\):

\[x^2 + 6x + y^2 + 8y = -9\]

Remember, to complete the square in a polynomial, we need to add:

\[\left(\frac{1}{2}b\right)^2\]

In this case, for \(x, b = 6\), and for \(y, b = 8\). Therefore, we add

\[\left(\frac{1}{2} \cdot 6\right)^2 - 9^2 = 9\]
\[
\left( \frac{1}{2} \right)^2 = 4^2 = 16
\]
to both sides of the equation:
\[x^2 + 6x + y^2 = 3y + 16 = -9 + 9 + 16\]
We then simplify and factor:
\[(x + 3)^2 + (y - 4)^2 = 16\]
Here, \( r = 4 \), which is positive, so this equation represents a circle with
Center: \((-3, -4)\)
Radius: 4

Exercise: #23
Determine whether the equation has a circle as its graph. If it does, give the center and radius. If not, describe the graph.
\[4x^2 + 4y^2 + 4x - 16y - 19 = 0\]
Solution:
We begin by moving the constant term to the other side of the equation and dividing by the coefficient of \( x^2 \):
\[4x^2 + 4x + 4y^2 - 16y = 19\]
\[x^2 + x + y^2 - 4y = \frac{19}{4}\]
Next, we add
\[\left( \frac{1}{2} \right)^2\]
for \( x \) and \( y \) for to both sides:
\[\left( \frac{1}{2} \right)^2 = \left( \frac{1}{2} \right)^2 = \frac{1}{4}\]
\[\left( \frac{1}{2} \right) \left( -4 \right)^2 = \left( -2 \right)^2 = 4\]
\[x^2 + x + \frac{1}{4} - 4y - 4 = \frac{19}{4} + \frac{1}{4} + 4\]
and factor to obtain:
\[\left( x + \frac{1}{2} \right)^2 + (y - 4)^2 = 9\]
This represents a circle with
Center: \((-1/2, 4)\)
Radius: 3

Exercise: #25
Determine whether the equation has a circle as its graph. If it does, give the center and radius. If not, describe the graph.
\[x^2 + y^2 + 2x - 6y + 14 = 0\]
Solution:
We go through the usual calculations and obtain the form:
\[(x + 1)^2 + (y - 3)^2 = -4\]
If we believe this is a circle, then \( r^2 = -4 \), which means that it would have an imaginary radius. This equation does not represent a circle.

Exercise: #27
Determine whether the equation has a circle as its graph. If it does, give the center and radius. If not, describe the graph.
\[x^2 + y^2 - 6x - 6y + 18 = 0\]
Solution:
We go through the usual calculations and obtain
\[(x-3)^2 + (y-3)^2 = 0\]

Since the radius is 0, this is the graph of the single point (3, 3).

\[\text{Exercise: #41}\]

Find the center-radius form of the equation of a circle with center (3, 2) and tangent to the x-axis. (Hint: A line tangent to a circle means touching it at exactly one point.)

Solution:

We are given the center (3, 2), and can visualize the distance from the center to the x-axis, which is 2, so the radius is also 2.

Therefore, the form is:

\[(x-3)^2 + (y-2)^2 = 4\]

\[\text{Exercise: #42}\]

Find the equation of a circle with center at (-4, 3), passing through the point (6, 8).

Solution:

Since we know the center and a point on the circle, we know that the distance from the center to that point must be the radius. We calculate:

\[d = \sqrt{(6 - (-4))^2 + (8 - 3)^2}\]

\[d = \sqrt{10^2 + 5^2}\]

\[d = \sqrt{106}\]

Therefore, the radius is \(\sqrt{106}\) right?

so \(r^2 = 106\), and the center-radius form is:

\[(x + 4)^2 + (y - 3)^2 = 106\]

\[\text{Exercise: #47}\]

Find all values of y such that the distance between (3, y) and (-2, 0) is 12.

Solution:

Here we can use the distance formula:

\[12 = \sqrt{(-2-3)^2 + (0-y)^2}\]

\[12 = \sqrt{25 + (0-y)^2}\]

\[144 = 25 + (0-y)^2\]

\[129 = 25 + 8y - y^2\]

\[y^2 - 8y + 129 = 0\]

This is just a quadratic equation, so we can use the quadratic formula with:

\[a = 1\]

\[b = -8\]

\[c = 129\]

This yields
\[ y = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(-30)}}{2} \]
\[ y = \frac{18 \pm \sqrt{324 + 152}}{2} \]
\[ y = \frac{18 \pm \sqrt{476}}{2} \]
\[ y = \frac{18 \pm 2\sqrt{119}}{2} \]
\[ y = 9 \pm \sqrt{119} \]
Functions, 2.3

Definition: A relation is a set of ordered pairs. We can think of a relation as any shape imaginable on the coordinate plane. Relations show how one quantity depends on another, e.g. a baby’s weight and a baby’s age may be closely related.

Definition: A function is a relation in which, for each distinct value of the first component of the ordered pairs (x), there is exactly one value of the second component (y). In other words, for every x there is one single y.

We can think of a function as a factory into which we put an x and retrieve a y. Every time we put in a particular x, we should receive the one y corresponding to that x. We say ‘y is a function of x’ to emphasize that y depends on x.

We have (some) freedom to choose any x to use as input. Therefore, this input is called the independent variable. The y-value depends on the x-value that was used as input. Hence, this output is called the dependent variable.

There are often some restrictions on the independent variable. The domain is the set of all real number x-values that produce real number y-values. In other words, which x’s give me valid y’s? We define the range as the set of all y-values that result after plugging in all the valid x; what do we obtain?

Exercise: #6

Decide whether each relation defines a function and give the domain and range.

\{(−12, 5), (−10, 3), (8, 3)\}

Solution:

We can tell that this is a function, because each x corresponds to one y:

\[
12 \rightarrow 5 \\
10 \rightarrow 3 \\
8 \rightarrow 3
\]

The domain is the set of x-values, or \{-12, -10, 8\}. The range is the set of y-values, or \{5, 3\}.

Exercise: #12

Decide whether each relation defines a function and give the domain and range.

![Diagram of a relation with ordered pairs (1, 10), (2, 15), (3, 19), (3, 15), (5, 27), (5, 19)]

Solution:

This is not a function, as 2 is associated with 15 and 19. This is one x that has two y’s.

The domain of the relation is \{1, 2, 3, 5\} and its range is \{10, 15, 19, 27\}.

Some relations contain an infinite number of points and are better graphed than listed. We would like to know their domains and ranges and we would like to know how to tell if graphed relations are functions.

Therefore, we have the vertical line test: if each vertical line intersects a graph in at most one point, then the graph is that of a function.
Exercise: #18

Use the given graph and determine domain, range and if it is a function.

Solution:

We note that every x-value corresponds to a y-value, so the domain is \( (-\infty, \infty) \), i.e. we can plug in any x and get a valid y.

We note, however, that the graph never rises higher than \( y = 4 \). Therefore, the range is \( (-\infty, 4] \).

Next, we note that every vertical line intersects the graph only once. Therefore, this is the graph of a function.

Exercise: #21

Use the given graph and determine domain, range and if it is a function.

Solution:

Because numerous vertical lines intersect the graph more than once, this is not a function. Take \( x = 0 \), for example.

The domain is the set of all x-values, \( (-\infty, 4] \), and the range the set of all y-values, \( (-\infty, 3] \).

Occasionally, we will just be given a formula, rather than a list of points or a graph. To find the domain given a formula, there are two restrictions to keep in mind. When dealing with functions, you are only interested in obtaining real numbers for your y-values. Therefore,

You are not able to divide by 0.
You cannot take the square root of a negative number.

Exercise: #27

Decide whether the relation below defines \( y \) as a function of \( x \). Give the domain and range.

\[ y = 2x - 5 \]

Solution:

In general, it may be difficult to tell whether a relation is a function. However, in this case, you can see easily that if you plug in a single x, you obtain a single y.

Alternatively, you can recognize that this is the graph of a line with slope 2. Therefore, this relation represents a function.

Now, because there is no possibility of division by 0 or taking the square root of a negative number, the domain must be \( (-\infty, \infty) \). The range is in general more difficult, but in this case we can view the graph again, realizing that the range must be \( (-\infty, \infty) \).
Exercise: #29

Decide whether $x + y < 3$ defines $y$ as a function of $x$. Give the domain.

Solution:

This is not a function. For instance, when we plug in $x = 0$, this implies that

$$
0 + y < 3
$$

$$
y < 3
$$

but an infinite number of $y$ satisfy this relation; e.g. $y = 2, 1, -5$, etc. For one $x$, we have an infinite number of $y$. Because we are not dividing or taking square roots, the domain is again $(-\infty, \infty)$.

Exercise: #36

Decide whether $y = \sqrt{2x}$ defines $y$ as a function of $x$. Give the domain and range.

Solution:

No matter which $x$-value we choose, there is only one $y$-value, so that this is a function. Now, in determining the domain, we may only take the square roots of non-negative numbers. In other words, we must have

$$7 - 2x \geq 0$$

$$-2x \geq -7$$

$$x \leq \frac{7}{2}$$

Therefore, the domain is $(-\infty, \frac{7}{2}]$. Now, we know that the square root of any non-negative number is non-negative, so the range is $[0, \infty)$.

Exercise: #38

Decide whether

$$y = \frac{-7}{x-5}$$

defines $y$ as a function of $x$. Give the domain and range.

Solution:

For every $x$ we plug in, there is one $y$, so this is a function.

To determine the domain, we note that we cannot have a 0 in the denominator. Therefore, to find the "holes" in the domain, we set the denominator equal to 0:

$$x - 5 = 0$$

which implies that $x = 5$

We must exclude this value from the domain. Therefore,

domain = $(-\infty, 5) \cup (5, \infty)$

To find the range in rational functions of this sort, there are three questions we ask:

Can $y$ be positive? Yes. For example, $x = 4 \rightarrow y = 7$.

Can $y$ be negative? Yes. For example, $x = 6 \rightarrow y = -7$.

Can $y$ be zero? No.

Therefore, the range is $(-\infty, 0) \cup (0, \infty)$.

Function Notation

Function Notation: When function $f$ is applied to an input $x$, we write $f(x)$ to represent the resulting output $y$. In other words, we write

$$y = f(x)$$

which is read "$y$ is equal to $f$ of $x$" and we have the vocabulary:

Name of function = $f$

Input = $x$

Output = $y$

Occasionally, we might be given an equation in $x$ and $y$ and want to rephrase it in function notation. To do this:

1. Solve for $y$.
2. Replace $y$ with $f(x)$.
Example: #64

Consider the equation \( x - 4y = 8 \). Rephrase this in function notation and calculate \( f(3) \).

Solution:

First, we solve for \( y \):

\[
x - 4y = 8
\]
\[
-4y = 8 - x
\]
\[
y = \frac{8 - x}{-4}
\]
\[
y = \frac{1}{4}x - 2
\]

We then replace \( y \) by \( f(x) \):

\[
f(x) = \frac{1}{4}x - 2
\]

Next, we calculate \( f(3) \):

\[
f(3) = \frac{1}{4}(3) - 2
\]
\[
f(3) = \frac{1}{4} \cdot 3 - 2
\]
\[
f(3) = \frac{3}{4} - 2
\]

We can find functions of real numbers, but we can also find functions of variables. For instance, let \( f(x) = -3x + 4 \). We can calculate:

\[
f(\frac{3}{2}) = -3\left(\frac{3}{2}\right) + 4
\]
\[
f(\frac{3}{2}) = 7 + 4
\]
\[
f(\frac{3}{2}) = 11
\]

\[
f(p) = -3(p) + 4
\]
\[
f(p) = -3p + 4
\]

\[
f(a + 4) = -3(a + 4) + 4
\]
\[
f(a + 4) = -3a - 12 + 4
\]
\[
f(a + 4) = -3a - 8
\]

Increasing, Decreasing, and Constant Functions

Suppose that a function \( f \) is defined over some interval \( I \). If \( x_1 \) and \( x_2 \) are points in that interval, we say that

a. \( f \) increases on \( I \) if, whenever \( x_1 < x_2 \),

\[
f(x_1) < f(x_2)
\]
b. \( f \) decreases on \( I \) if, whenever \( x_1 < x_2 \),
\[
f(x_1) > f(x_2).
\]

\[
\begin{array}{c}
\hline
\text{Graph of decreasing function.}
\end{array}
\]

\[\text{c.} \quad f \text{ is constant on } I \text{ if, for every } x_1 \text{ and } x_2,
\]
\[
f(x_1) = f(x_2).
\]

\[\text{Example: #78}
\]

Determine the intervals of the domain for which each function is
\[\text{a. increasing,}
\]
\[\text{b. decreasing, and}
\]
\[\text{c. constant.}
\]

\[
\begin{array}{c}
\hline
\text{Graph of function with intervals marked.}
\end{array}
\]

\[\text{Solution:}
\]
\[\text{The function is increasing on } (-\infty, 1], \text{ decreasing on } [1, \infty), \text{ and constant on } [1, 4].
\]

\[\text{Example: #78}
\]

Determine the intervals of the domain for which each function is
\[\text{a. increasing,}
\]
\[\text{b. decreasing, and}
\]
\[\text{c. constant.}
\]

\[
\begin{array}{c}
\hline
\text{Graph of function with intervals marked.}
\end{array}
\]

\[\text{Solution:}
\]
\[\text{The function is increasing on } (-\infty, 1], \text{ decreasing on } [1, \infty), \text{ and constant on } [1, 4].
\]
Exercise: #8.4

A ball is thrown straight up into the air. The function defined by \( y = h(t) \) in the graph gives the height of the ball (in feet) at \( t \) seconds.

![Graph of a ball's height over time](image)

a. What is the height of the ball at 2 seconds? Each tick mark is 16 units, so the height is 256-16 = 240 ft.
b. When will the height be 192 ft? At 1 and 6 seconds.
c. During what time intervals is the ball going up? Down? Up: \([0, 3]\), down: \([3, 7]\).
d. How high does the ball go, and when does the ball reach its maximum height? The ball reaches a maximum height of 256 feet in 3 seconds.
e. After how many seconds does the ball hit the ground? The ball hits the ground in 7 seconds.
Linear Functions, 2.4

Functions are infinite in number, but there is one family of functions that shows up everywhere. Any function of the form

\[ f(x) = ax + b \]

where \( a, b \) are numbers is called a linear function. Linear functions, when graphed, are always lines.

Each linear function has a slope, \( m \), which can be calculated from any two points \((x_1, y_1)\) and \((x_2, y_2)\) by the formula

\[
m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}
\]

where \( \Delta x = 0 \).

If the slope is positive, the graph is of a rising line.

If the slope is positive, the graph is a falling line.

A slope of 0 indicates a horizontal line.

An undefined slope indicates a vertical line.
Example: #36

A line passes through the points \((5, -3)\) and \((1, -7)\). Find its slope.

Solution:

\[
m = \frac{-7 - (-3)}{1 - 5} = \frac{-4}{-4} = 1
\]

Example: #46

Find the slope of the line \(y = 2x - 4\).

Solution:

We aren’t given any points so we came up with our own. We choose \(x = 0, 1, 2\) and calculate the \(y\)-values.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-4</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>-2</td>
</tr>
</tbody>
</table>

We calculate the slope between the points \((0, -4)\) and \((2, 0)\):

\[
m = \frac{0 - (-4)}{2 - 0} = \frac{4}{2} = 2
\]

Therefore, the line has a slope of 2. The slope of the line should be the same, no matter which points you may have selected. Check this using the other point!

Example: #17

Graph \(f(x) = -4\)

Solution:

This is just the line \(y = -4\). This is called a constant function.

All horizontal lines are functions, because they pass the vertical line test. Vertical lines always fail the vertical line test, however, and are never functions.

Exercise: #54

Graph the line passing through \((-2, 8)\) and having the slope \(m = -1\). Plot two points on the line.

Solution:

We begin at the point \((-2, 8)\) and know that

\[
m = \frac{\text{rise}}{\text{run}} = -1
\]

Therefore, for every unit we move to the right (run = 1), we move down one unit (rise = -1). We end up with the graph
The slope also gives information about the average rate of change in y per unit change in x.

Example: #76
When introduced in 1997, a DVD player sold for about $500. In 2007, a DVD player could be purchased for $90. Find and interpret the average rate of change in price per year.

Solution:
We have the two points (1997, 500) and (2007, 90). The cost of the DVD player depends on the year, so the cost is the dependent variable, y. Then, the rate of change is the slope, so we calculate

\[ m = \frac{500 - 90}{1997 - 2007} \]
\[ m = 410 \]
\[ m = -41 \]

This means that the cost dropped $41 dollars per year (y per x).
Textbook Section 2.5
Equations of Lines

Objectives:

- The student will be able to find equation of a line
- The student will be able to find slope, \( y \)-intercept, and graph line
- The student will be able to write equation of a line given a graph
- The student will be able to write equation in slope intercept and standard forms
- The student will be able to use slope to decide if three points are collinear
- The student will be able to solve modeling problems

Key Concepts

Point-Slope Form:

\[ m = \text{slope} \]
\[ (x_1, y_1) = \text{any point on the line} \]

Equation of line: \[ y - y_1 = m(x - x_1) \]

Slope-Intercept Form:

\[ m = \text{slope} \]
\[ (0, b) = \text{y-intercept point} \]

Equation of line: \[ y = mx + b \]

Horizontal and Vertical Lines:

An equation of the horizontal line through the point \((a, b)\) is \(y = b\).
An equation of the vertical line through the point \((a, b)\) is \(x = a\).

Parallel Lines:

Two distinct non-vertical lines are parallel if and only if they have the same slope.

Perpendicular Lines:

Slopes have a product of \(-1\).
Slopes are negative reciprocals.

Textbook Section 2.6
Graphs of Basic Functions

Objectives:

- The student will be able to determine intervals of continuity
- The student will be able to evaluate and graph piece-wise functions
- The student will be able to find the rules (the equations) for a piecewise function
- The student will be able to graph a greatest integer function
- The student will be able to solve modeling problems

Key Concepts

Continuity:

A function is continuous over an interval of its domain if the graph can be sketched without lifting the pencil from the paper.
Basic Graphs:
Be able to recognize the shapes of these graphs.
Be familiar with the domain and range for each.

Type 1) Identity Function: \( f(x) = x \)
Type 2) Squaring Function: \( f(x) = x^2 \)
Type 3) Cubing Function: \( f(x) = x^3 \)
Type 4) Square Root Function: \( f(x) = \sqrt{x} \)
Type 5) Cube Root Function: \( f(x) = \sqrt[3]{x} \)
Type 6) Absolute Value Function: \( f(x) = |x| \)
Type 7) Greatest Integer Function: \( f(x) = \lfloor x \rfloor \)
This function pairs every real number \( x \) with the greatest integer less than or equal to \( x \).

Textbook Section 2.7
Graphing Techniques

Objectives

- The student will be able to graph functions using transformations, such as translation, stretch, shrink, and reflect
- The student will be able to describe how changes to the equation of a basic function will affect the graph
- The student will be able to find the equation of a given graph
- The student will be able to analyze symmetry using the graph or the equation
- The student will be able to determine whether functions are even, odd, or neither

Key Concepts

Translation:
\[ y = f(x) + k \]
is translated \( k \) units up.
\[ y = f(x) - k \]
is translated \( k \) units down.
\[ y = f(x + h) \]
is translated \( h \) units to the left.
\[ y = f(x - h) \]
is translated \( h \) units to the right.

Vertical Stretching and Shrinking:
\[ y = a \cdot f(x) \]
For \( 0 < |a| < 1 \) → vertical shrink (compressed)
For \( |a| > 1 \) → vertical stretch

A reflection forms a mirror image of a graph across a line.
The graph of \( y = -f(x) \) is reflected across the x-axis.
The graph of \( y = f(-x) \) is reflected across the y-axis.

Y-axis symmetry:
Even function if \( f(-x) = f(x) \)
Reflection of the graph over the \( y \)-axis yields the same picture.
Replacement of \( x \) with \(-x\) results in an equivalent equation.

X-axis symmetry:
Reflection of the graph over the \( x \)-axis yields the same picture.
Replacement of \( y \) with \(-y\) results in an equivalent equation.

Origin symmetry:
Odd function if \( f(-x) = -f(x) \)
Rotation about the origin \((0,0)\) yields the same picture.
Replacement of \( x \) with \(-x\) and \( y \) with \(-y\) results in an equivalent equation.
Handouts

Some of the files you are about to view/download are PDF files. If you do not have Adobe Acrobat installed on your system, you can download the free Adobe Acrobat Reader at http://www.adobe.com/products/acrobat/alternate.html

- Equations of Lines Handout
- Basic Graphs Handout
- Even and Odd Functions Handout
- Greatest Integer Handout

Online Homework and Quiz Assignments

After reviewing the Key Concepts, log into MyLabsPlus and begin your homework and quiz for this week. Then go to www.ucf.mylabsplus.com and begin working on your assignments.
Equations of Lines, 2.5

Lines can be written in a variety of useful forms. We have the standard form

\[ Ax + By = C \]

where \(A\), \(B\), and \(C\) are integers and \(A > 0\). There is also the point-slope form. If we have \(m = \text{slope}\) and \((x_1, y_1)\), any point on the line, then we may write

\[ y - y_1 = m(x - x_1) \]

The point-slope form can be used to easily find equations of lines.

Example: #18

Find the equation of the line passing through \((8, -1)\) and \((4, 3)\).

Solution:

We first calculate the slope:

\[ m = \frac{3 - (-1)}{4 - 8} \]
\[ m = \frac{4}{-4} \]
\[ m = -1 \]

Second, we choose one of the points. Here, we use \((8, -1)\). We plug these into the point-slope form:

\[ y - (-1) = -1(x - 8) \]
\[ y + 1 = -x + 8 \]
\[ y = -x + 7 \]

There is also the slope-intercept form:

\[ y = mx + b \]

where

\(m = \text{slope}\)

\((0, b) = y\text{-intercept}\)

Example: #16

Find the equation of the line with \(x\)-intercept \(-2\), \(y\)-intercept \(4\).

Solution:

We have the two points \((-2, 0)\) and \((0, 4)\), so we can find the slope:

\[ m = \frac{4-0}{0-(-2)} \]
\[ m = \frac{4}{2} \]
\[ m = 2 \]

Now, we also have the \(y\)-intercept, \((0, 4)\), so \(b = 4\). Therefore, the line is given by \(y = 2x + 4\). We can write this in standard form:

\[ 2x - y = -4 \]

Example: #32

Give the slope and \(y\)-intercept of the line \(y = -2x + 7\) and graph it.

Solution:

Since this line is in slope-intercept form, we can read directly from the equation that \(m = -2\), so the slope is negative and we should expect a falling line. The \(y\)-intercept is \((0, 7)\).
Example: #34
Convert $2x + 3y = 16$ to slope-intercept form and graph.

Solution:

We solve for $y$:

$\begin{align*}
2x + 3y &= 16 \\
3y &= -2x + 16 \\
y &= -\frac{2}{3}x + \frac{16}{3}
\end{align*}$

Therefore:

$m = -\frac{2}{3}$

$y$-intercept = $\left(0, \frac{16}{3}\right)$

Parallel lines have the same slope. Perpendicular lines have slopes that are negative reciprocals; in other words, the product of the two slopes is $-1$.

Example: #50
Write an equation in standard form for a line that passes through $(2, 0)$ and is perpendicular to $8x - 3y = 7$.

Solution:

We already know that the line passes through $(2, 0)$, so we need its slope. First, we find the slope of $8x - 3y = 7$:

$\begin{align*}
8x - 3y &= 7 \\
-3y &= -8x + 7 \\
y &= \frac{8}{3}x - \frac{7}{3}
\end{align*}$

slope of this line = $\frac{8}{3}$

Now, our new line is perpendicular to this line, so the slope of the new line is the negative reciprocal of this slope:

$m = -\frac{8}{3}$

$= -\frac{3}{8}$
Since we have a point and the slope, we can use the point-slope form to find the line:

\[ y - y_1 = m(x - x_1) \]
\[ y - 0 = \frac{-3}{8} (x - (-2)) \]
\[ y = \frac{-3}{8} (x + 2) \]
\[ 8y = -3(x + 2) \]
\[ 8y = -3x - 6 \]
\[ 3x + 8y = -6 \]
Graphs of Basic Functions, 2.6

Continuity is an important concept in mathematics; here we're going to give an informal definition. A function is continuous over an interval of its domain if its hand-drawn graph can be sketched without lifting the pencil from the paper.

Example: #13)

Determine the intervals of the domain over which the function graphed here is continuous.

Solution: Beginning with \( x = 0 \), we can draw the entirety of the graph to the right without having to lift the pencil from the paper. Therefore, the graph is continuous on the interval \([0, \infty)\).

Example: #15)

Determine the intervals of the domain over which the function graphed here is continuous.

Solution: We don't really begin at \(-\infty\), but we can think about beginning far on the left and continuing drawing the graph until we get to \( x = 1 \), at which point we have to lift up the pencil. Therefore, the graph is continuous for \( x < 1 \). We begin again at \( x = 1 \) and can draw towards \( \infty \). Therefore, the function is continuous on \((-\infty, 1) \cup \{1, \infty\}\).

Basic Graphs:

There are some basic functions and graphs that you will want to be familiar with.

Type 1) Identity Function

\[ f(x) = x \]

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-2</td>
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<tr>
<td>-1</td>
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<td>1</td>
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<tr>
<td>2</td>
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</tr>
</tbody>
</table>

Domain = \([-\infty, \infty]\)

Range = \((-\infty, \infty)\)

Type 2) Squaring Function (parabola):

\[ f(x) = x^2 \]
Type 3) Cubing Function:
\[ f(x) = x^3 \]

Type 4) Square Root Function:
\[ f(x) = \sqrt{x} \]

Type 5) Cube Root Function:
\[ f(x) = \sqrt[3]{x} \]
Domain = \([-\infty, \infty]\)
Range = \([-\infty, \infty]\)

**Piecewise Defined Function:**
A function defined by different rules over different intervals of its domain

**Type 6) Absolute Value Function:**

\[ f(x) = |x| \]

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
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</thead>
<tbody>
<tr>
<td>-2</td>
<td>2</td>
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<tr>
<td>-1</td>
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<td>2</td>
</tr>
</tbody>
</table>

Domain = \([-\infty, \infty]\)
Range = \([0, \infty]\)

**Type 7) Greatest Integer Function:**
(also called a step function)

This function pairs every real number \(x\) with the greatest integer less than or equal to \(x\).

\[ f(x) = \lfloor x \rfloor \]

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-2</td>
</tr>
<tr>
<td>-1.5</td>
<td>-2</td>
</tr>
<tr>
<td>-1.01</td>
<td>-2</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>-0.99</td>
<td>-1</td>
</tr>
<tr>
<td>-0.01</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
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<tr>
<td>0.01</td>
<td>0</td>
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<tr>
<td>0.5</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1.99</td>
<td>1</td>
</tr>
<tr>
<td>1.001</td>
<td>1</td>
</tr>
</tbody>
</table>

**Exercise: #20)**

For the piecewise-defined function find \(f(-5), f(-1), f(3)\).

\[ f(x) = \begin{cases} 
-2x & \text{if } x < -3 \\
3x-1 & \text{if } -3 \leq x \leq 2 \\
-4x & \text{if } x > 2 
\end{cases} \]

Solution: In calculating \(f(-5)\), we know that \(x = -5\), so we are dealing with the first rule, since \(-5 < -3\). Therefore,

\[ f(-5) = -2[-5] = 10 \]

In calculating \(f(-1)\), we know that \(x = -1\), so we are dealing with the second rule, since \(-3 \leq -1 \leq 2\). Therefore,

\[ f(-1) = 3[-1]-1 = -3 - 1 = -4 \]

In calculating \(f(3)\), we know that \(x = 3\), so we are dealing with the third rule, since \(3 > 2\). Therefore,

\[ f(3) = -4[3] = -12 \]
Exercise: #28)

Graph the following piecewise function:

\[ f(x) = \begin{cases} 
-2x & \text{if } x < -3 \\
3x - 1 & \text{if } -3 \leq x \leq 2 \\
-4x & \text{if } x > 2 
\end{cases} \]

Solution: We note that on each interval, the graph is just a line. Therefore, while we're graphing less than -3, we can use points from -2x. We can break the plot into three intervals and try points:

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>x</th>
<th>y</th>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3.001</td>
<td>6.002</td>
<td>-3</td>
<td>-10</td>
<td>-0.001</td>
<td>-6.004</td>
</tr>
<tr>
<td>-4</td>
<td>8</td>
<td>0</td>
<td>-1</td>
<td>3</td>
<td>-12</td>
</tr>
<tr>
<td>-5</td>
<td>10</td>
<td>2</td>
<td>5</td>
<td>4</td>
<td>-16</td>
</tr>
</tbody>
</table>

Exercise: #47)

Assume that postage rates are 41 cents for the first ounce, plus 17 cents for each additional ounce, and that each letter carries one 41 cent stamp and as many 17 cent stamps as necessary. Graph the function \( f \) that models the number of stamps on a letter weighing \( x \) ounces over the interval \((0, 6]\).

Solution:

<table>
<thead>
<tr>
<th>Ounces</th>
<th>Postage</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 1]</td>
<td>$0.41</td>
</tr>
<tr>
<td>(1, 2]</td>
<td>$0.58</td>
</tr>
<tr>
<td>(2, 3]</td>
<td>$0.75</td>
</tr>
<tr>
<td>(3, 4]</td>
<td>$0.92</td>
</tr>
<tr>
<td>(4, 5]</td>
<td>$1.09</td>
</tr>
</tbody>
</table>
Graphing Techniques, 2.7

We have the basic graphs of the previous section and can obtain the graphs of many more functions by modifying those slightly. The first modification is the translation or shift: movement of graph up, down, left, or right.

Vertical Shift:
• The graph of \( y = f(x) + k \) is translated \( k \) units up.
• The graph of \( y = f(x) - k \) is translated \( k \) units down.
• Amount added or subtracted outside basic function.

Horizontal Shift:
• The graph of \( y = (x + h) \) is translated \( h \) units to the left.
• The graph of \( y = (x - h) \) is translated \( h \) units to the right.
• Amount added or subtracted inside basic function

For example, we recall our basic function \( y = x^2 \), which has the following graph:

![Graph of \( y = x^2 \)](image)

We can consider the two kinds of shifts:

- **Shifted DOWN by 2**
  - \( y = x^2 - 2 \)
  - ![Table and Graph](image)

- **Shifted UP by 3**
  - \( y = x^2 + 3 \)
  - ![Table and Graph](image)

- **Shifted RIGHT by 2**
  - \( y = (x - 2)^2 \)
  - ![Table and Graph](image)

- **Shifted LEFT by 3**
  - \( y = (x + 3)^2 \)
  - ![Table and Graph](image)

Vertical Stretching and Shrinking:

\[ y = a \cdot f(x) \]

Let \( f(x) = x^2 \) be our basic function, which we graph below in black. We can consider \( y = 2x^2 \) and \( y = \frac{1}{3}x^2 \).
We conclude that if you multiply by an \( a \) which satisfies

\[0 < |a| \leq 1\]

then you compress the graph. If you multiply by an \( a \) which satisfies

\[|a| > 1\]

then this represents a vertical stretch of your graph.

**NOTE:** If \((x,y)\) is on \(y = f(x)\), then \((x, ay)\) is on \(y = a \cdot f(x)\). For example, if your original graph has \((2,4)\) and you consider the stretched graph \(3f(x)\), then the new graph has the point \((2,12)\).

**Definition:** A reflection forms a mirror image of a graph across a line. For example, let \(f(x) = \sqrt{x}\) be our basic function, graphed below in black.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
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</thead>
<tbody>
<tr>
<td>-1</td>
<td>und</td>
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<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
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</tbody>
</table>

We can consider the two reflections \(-f(x)\) and \(f(-x)\):

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
<th>(x)</th>
<th>(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
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<td>4</td>
<td>-2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>-3</td>
<td>1</td>
<td>und</td>
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</tbody>
</table>

**Conclusions**

We conclude that the graph of \(y = -f(x)\) is reflected across the \(x\)-axis. The graph of \(y = f(-x)\) is reflected across the \(y\)-axis.

**NOTE:** If \((x,y)\) lies on the graph of \(y = f(x)\), then \((-x,y)\) lies on the graph of \(y = -f(x)\) and \((-x,y)\) lies on the graph of \(y = f(-x)\).

**Example:**

Graph the function

\[y = \sqrt{-x}\]

What changes to the basic graph should we expect?
Solution:

This is a reflection across the $y$-axis of the basic graph

$$y = \frac{1}{x}$$

The basic graph is in black, while this new graph is in red.

Example:

Graph $y = |x + 3| + 2$.

Solution:

The basic graph here is the absolute value graph. This is a vertical shift up of 2 units and a horizontal shift left of 3 units.

Example:

Describe the transformations and give the equation for the graph.

The basic graph is the square root. This is a reflection over the $x$-axis, a shift left by 4, and a shift up by 2.
The equation is therefore

$$y = -\sqrt{x + 4} + 2$$

Symmetry can help us further understand graphs. There are three types of symmetries we're concerned with:

Y-axis symmetry: The graph of an equation is symmetric with respect to the $y$-axis if

**Graph Test:**
Reflection of the graph over the $y$-axis yields the same picture.

**Equation Test:**
Replacement of $x$ with $-x$ results in an equivalent equation.
A function is called an even function if it is symmetric with respect to the y-axis.

**X-axis symmetry:** The graph of an equation is symmetric with respect to the x-axis if

**Graph Test:**
Reflection of the graph over the x-axis yields the same picture.

**Equation Test:**
Replacement of y with − y results in an equivalent equation.

Only one function is symmetric with respect to the x-axis. Which one?

**Origin symmetry:** The graph of an equation is symmetric with respect to the origin if

**Graph Test:**
Rotation of the graph about the origin yields the same picture.

**Equation Test:**
Replacement of both x with − x and y with − y results in an equivalent equation.

A function is called an odd function if \( f(-x) = -f(x) \) for all \( x \) in the domain of \( f \).

Keep in mind that it is possible for an equation to have no symmetries. It is possible for a function to not be even or odd.

**Example: #28)**

Consider the symmetries (if any) of \( y = x^3 - x \)

**Solution:**

We usually won't have access to the graph, but here we include it just for confirmation of what we will find below. Which of the above symmetries, if any, should we expect?

![Graph of y = x^3 - x](image)

To test for y-axis symmetry, we replace \( x \) by − \( x \). If this relation is symmetric with respect to the y-axis, the relation should stay unchanged. We calculate:

\[
\begin{align*}
y &= (-x)^3 - (-x) \\
y &= -x^3 + x
\end{align*}
\]

This is not the same relation, so this relation is not symmetric with respect to the y-axis.

To test for x-axis symmetry, we replace \( y \) by − \( y \). If this relation is symmetric with respect to the x-axis, the relation should stay unchanged. Clearly, it does not.

To test for origin symmetry, we replace \( x \) by − \( x \) and \( y \) by − \( y \). If this relation is symmetric with respect to the y-axis, the relation should stay unchanged. We calculate:

\[
\begin{align*}
(-y) &= (-x)^3 - (-x) \\
-y &= -x^3 + x
\end{align*}
\]

Now, if we multiply both sides of the equation by −1, we obtain

\[
\begin{align*}
(-1)(-y) &= (-1)(-x^3 + x) \\
y &= x^3 - x
\end{align*}
\]

This is what we started with. Therefore, this relation is symmetric with respect to the origin.

**Example: #31)**

Decide whether \( f(x) = -x^3 + 2x \) is even, odd, or neither.

**Solution:**

We calculate

\[
\begin{align*}
f(-x) &= -(-x)^3 + 2(-x) \\
&= -x^3 - 2x \\
&= -[x^3 - 2x] \\
&= -f(x)
\end{align*}
\]

This is an odd function and symmetric with respect to the origin.
Example: #39)

Decide whether \( f(x) = x^3 - x + 9 \) is even, odd, or neither.

Solution:

We check

\[
f(-x) = (-x)^3 - x + 9 \\
= -x^3 + x + 9 \\
\ne \ne f(x)
\]

The function is not even.

\[
-f(x) = - (x^3 - x + 9) \\
= -x^3 + x - 9 \\
\ne \ne f(-x)
\]

The function is not odd. Therefore, the function is neither.
Various form of linear equations

**Slope-Intercept form**

\[ y = mx + b, \quad m \text{ is the slope and } b \text{ is the y-intercept.} \]

**Standard Form**

\[ Ax + By = C \]

**Point-Slope Form**

\[ y - y_1 = m(x - x_1), \quad m \text{ is the slope and the line passes through the point } (x_1, y_1) \]

**Converting Standard Form to Slope-Intercept Form**

**Exercise 2.5.33**

Give the slope and y-intercept of the line

\[ 4x - y = 7 \quad \text{Line Is in Standard Form} \]

\[ -y = -4x + 7 \quad \text{Subtract } 4x \]

\[ y = 4x + (-7) \quad \text{Divide by -1} \]

The equation is in the slope-intercept form \( y = mx + b \).

The slope is \(4\) and the y-intercept is \(-7\).

**Exercise 2.5.34**

Give the slope and y-intercept of the line

\[ 2x + 3y = 16 \quad \text{Line Is in Standard Form} \]

\[ 3y = -2x + 16 \quad \text{Subtract } 2x \]

\[ y = -\frac{2}{3}x + \frac{16}{3} \quad \text{Divide by 3} \]

The equation is in the slope-intercept form \( y = mx + b \).

The slope is \(-\frac{2}{3}\) and the y-intercept is \(\frac{16}{3}\).
Converting Point-Slope Form to Standard Form

Exercise 2.5.6

Find an equation in standard form of the line having the given slope and containing the given point.

\( m = -2, \ (1,5) \)

Here \( x_1 = 1, y_1 = 5 \) and \( m = -2 \) and plug them in the point-slope form

\[
y - y_1 = m(x - x_1)
\]

Point-slope form

\[
y - 5 = -2(x - 1)
\]

\( x_1 = 1, y_1 = 5, \ m = -2 \)

\[
y - 5 = -2x + 2
\]

Be careful with signs.

\[
y = -2x + 7
\]

Add 5

This is the equation of the line in the slope-intercept form with slope \(-2\) and the y-intercept 7.

But we are asked to find an equation in standard form which is \( Ax + By = C \)

\[
y = -2x + 7
\]

\[
y + 2x = 7
\]

Add 2x

Now the above equation is in the standard form.

Finding the equation of the line given two points

Example 4 Section 2.5

Find an equation of the line through \((1,1)\) and \((2,4)\).

First we will find the slope of this equation.

\[
m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{where} \ (x_1,y_1) \ \text{and} \ (x_2,y_2) \ \text{are points on the line and} \ \Delta x \neq 0
\]

Pick any one of \((1,1)\), \((2,4)\) to be \((x_1,y_1)\) and the other will be \((x_2,y_2)\). It makes no difference which point is \((x_1,y_1)\) or \((x_2,y_2)\) however, be consistent.

Let \( x_1 = 1, \ y_1 = 1, \ and \ x_2 = 2, \ y_2 = 4 \)

So \[
m = \frac{(4-1)}{(2-1)} = \frac{3}{1} = 3.
\]
Now you can substitute 3 for \( m \) in slope-intercept form \( y = mx + b \) and choose any one of the given points say (1,1) and can find the y-intercept \( b \).

\[
\begin{align*}
y &= mx + b & \text{Slope-intercept form} \\
y &= 3x + b & m = 3 \\
1 &= 3(1) + b & x = 1, y = 1 \\
1 &= 3 + b & \text{Solve for } b \\
b &= -2
\end{align*}
\]

So the slope-intercept form is \( y = 3x - 2 \)

**Equations of Vertical and Horizontal Lines**

**Horizontal Line**

The slope of a horizontal Line is 0.

An equation of the horizontal line through the point \((a, b)\) with \( m = 0 \) is \( y = b \)

We can check that by substituting the point and the slope in the point-slope form

\[
\begin{align*}
y - y_1 &= m(x - x_1) & \text{Point-slope form} \\
y - b &= 0(x - a) & x_1 = a, y_1 = b, m = 0 \\
y &= b
\end{align*}
\]

**Exercise 2.5.20**

Write an equation for the horizontal line that passes through \((-8, -2)\). Give an answer in slope-intercept form.

As the horizontal line passes through \((-8, -2)\) so \( a = -8 \) and \( b = -2 \)

So the equation of horizontal line is \( y = -2 \)
**Vertical Line**

The Slope of a Vertical Line is undefined.

An equation of the Vertical line through the point \((a, b)\) is \(x = a\)

**Exercise 2.5.17**

Write an equation for the vertical line that passes through \((-6, 4)\). Give an answer in slope-intercept form.

As the vertical line passes through \((-6, 4)\) so \(a = -6\) and \(b = 4\)

So the equation of Vertical line is \(x = -6\)

**Parallel Lines**

Two distinct non-vertical lines are parallel if and only if they have the same slope.

**Example**

The lines \(y = 3x - 2\) and \(y = 3x + 12\) are parallel since their slopes, which is 3, are same.

**Perpendicular Lines**

Two lines, neither of which is vertical, are perpendicular if and only if their slopes have a product of -1. Thus, the slopes of perpendicular lines, neither of which are verticals, are negative reciprocals.

**Example**

The Lines \(y = 2x + 11\) and \(y = -\frac{1}{2}x + 17\) are perpendicular.

Slope of \(y = 2x + 11\) is 2.

Slope of \(y = -\frac{1}{2}x + 17\) is \(-\frac{1}{2}\).

The product of their slopes \(2 \left(-\frac{1}{2}\right) = -1\)

So the lines \(y = 2x + 11\) and \(y = -\frac{1}{2}x + 17\) are perpendicular.
**Exercise 2.5.47**

Write an equation in slope-intercept form for the line that passes through \((-1, 4)\) and is parallel to \(x + 3y = 5\).

First we will find the slope of \(x + 3y = 5\) by converting this standard form equation to slope-intercept form.

\[
\begin{align*}
x + 3y &= 5 \quad \text{Line Is in Standard Form} \\
3y &= -x + 5 \quad \text{Subtract } x \\
y &= -\frac{1}{3}x + \frac{5}{3} \quad \text{Divide by 3}
\end{align*}
\]

The slope \(m = -\frac{1}{3}\).

Line parallel to \(x + 3y = 5\) will have the same slope so now we have to find an equation of the line that passes through \((-1, 4)\) with slope \(m = -\frac{1}{3}\).

We will use the point-slope form to find the equation

\[
y - y_1 = m(x - x_1) \quad \text{Point-slope form}
\]

\[
y - 4 = -\frac{1}{3}(x - (-1)) \quad x_1 = -1, y_1 = 4, m = -\frac{1}{3}
\]

\[
y - 4 = -\frac{1}{3}(x + 1) \quad \text{Be careful with signs.}
\]

\[
y - 4 = -\frac{1}{3}x - \frac{1}{3} \quad \text{Solve for } y
\]

\[
y = -\frac{1}{3}x + \frac{11}{3} \quad \text{Add 4}
\]

\(y = -\frac{1}{3}x + \frac{11}{3}\) is the equation of the line that passes through \((-1, 4)\) and is parallel to \(x + 3y = 5\).
**Exercise 2.5.49**

Write an equation in slope-intercept form for the line that passes through \((1, 6)\) and is perpendicular to \(3x + 5y = 1\).

First we will find the slope of \(3x + 5y = 1\) by converting this standard form equation to slope-intercept form.

\[
3x + 5y = 1 \quad \text{Line Is in Standard Form}
\]

\[
5y = -3x + 1 \quad \text{Subtract} \ 3x
\]

\[
y = -\frac{3}{5}x + \frac{1}{5} \quad \text{Divide by 5}
\]

The slope \(m = -\frac{3}{5}\).

Line perpendicular to \(3x + 5y = 1\) will have the slope negative reciprocal of \(-\frac{3}{5}\).

\[
m_\perp = -\frac{1}{-\frac{3}{5}} = \frac{5}{3}
\]

So now we have to find an equation of the line that passes through \((1, 6)\) with slope \(m_\perp = \frac{5}{3}\).

We will use the point-slope form to find the equation.

\[
y - y_1 = m(x - x_1) \quad \text{Point-slope form}
\]

\[
y - 6 = \frac{5}{3}(x - 1) \quad x_1 = 1, y_1 = 6, m = \frac{5}{3}
\]

\[
y - 6 = \frac{5}{3}x - \frac{5}{3} \quad \text{Solve for} \ y.
\]

\[
y = \frac{5}{3}x + \frac{13}{3} \quad \text{Add} \ 6
\]

\[
y = \frac{5}{3}x + \frac{13}{3}\] is the equation of the line that passes through \((-1, 4)\) and is perpendicular to \(3x + 5y = 1\).
Type 1) **Identity Function:**

\[ f(x) = x \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-2</td>
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<tr>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
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<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

Domain = \((-\infty, \infty)\) \hspace{1cm} Range = \((-\infty, \infty)\)

Type 2) **Squaring Function:**

\[ f(x) = x^2 \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>4</td>
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<td>1</td>
<td>1</td>
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<tr>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

Domain = \((-\infty, \infty)\); \hspace{1cm} Range = \([0, \infty)\)
Type 3) Cubing Function: \[ f(x) = x^3 \]

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>−2</td>
<td>−8</td>
</tr>
<tr>
<td>−1</td>
<td>−1</td>
</tr>
<tr>
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<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
</tr>
</tbody>
</table>

Domain = \((−\infty, \infty)\); Range = \((−\infty, \infty)\)

Type 4) Square Root Function: \[ f(x) = \sqrt{x} \]

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>−1</td>
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</tr>
<tr>
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</tr>
<tr>
<td>1</td>
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</tr>
<tr>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
</tr>
</tbody>
</table>

Domain = \([0, \infty)\); Range = \([0, \infty)\)
Type 5) Cube Root Function:

\[ f(x) = \sqrt[3]{x} \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-8)</td>
<td>(-2)</td>
</tr>
<tr>
<td>(-1)</td>
<td>(-1)</td>
</tr>
<tr>
<td>(0)</td>
<td>(0)</td>
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<tr>
<td>(1)</td>
<td>(1)</td>
</tr>
<tr>
<td>(8)</td>
<td>(2)</td>
</tr>
</tbody>
</table>

Domain = \(-\infty, \infty\); Range = \(-\infty, \infty\)

Type 6) Absolute Value Function:

\[ f(x) = |x| \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-2)</td>
<td>(2)</td>
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<tr>
<td>(-1)</td>
<td>(1)</td>
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<td>(1)</td>
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<td>(2)</td>
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</tbody>
</table>

Domain = \(-\infty, \infty\); Range = \([0, \infty)\)
Odd and Even Functions

The properties you need to memorize about odd and even functions are as follows:
For odd functions, we have the property that \( f(-x) = -f(x) \).
For even functions, we have the property that \( f(-x) = f(x) \).

But there are more useful interpretations of the above equations. Consider the requirement for a function to be odd: \( f(-x) = -f(x) \). The left side of the equation says that when we take opposite values for \( x \), we get the opposite of the corresponding \( y \)-value. The MyLabsPlus explanation is that you multiply the coordinates by -1, and you’ll still be on the graph. But physically this means that if you take the function and fold it over the \( y \)-axis, and then the \( x \)-axis, your graph will be on top of itself. This doesn’t mean the graph is symmetric to the \( y \)-axis OR the \( x \)-axis. It means the graph is symmetric with respect to the origin.

Example: Determine if the following function is odd or even: \( y = x^3 \).
The first thing we do to find out if a function is odd or even is to replace \( x \) with \((-x)\), and simplify the function: \( y = (-x)^3 = -(x)(-x)(-x) = -x^3 \). Is this equal to \( f(x) \) or \(-f(x)\)? It is equal to \(-f(x)\), since \(-f(x) = -(x^3) = -x^3\). So what do we say? We say that since \( f(-x) = -f(x) \), the function is odd. If you look at the graph of \( y = x^3 \), notice if you fold it over the \( y \)-axis, and then over the \( x \)-axis, the graph is folded back on top of itself. This means \( y = x^3 \) is symmetric with respect to the origin.

Example: Determine if the graph of \( f(x) = |x| + 2 \) is symmetric to the \( x \)- or \( y \)-axis.
The first thing we do is replace with \( x \) with \(-x\) and simplify. We have \( f(-x) = |-x| + 2 = |x| + 2 \) (remember that \(-a = |a|\)). So now, is this equal to \( f(x) \) or \(-f(x)\)? Well, it is equal to \( f(x) \), so we say \( f(-x) = f(x) \), and \( f(x) \) is even. If you shift the graph of \( y = |x| \) up two units, notice that it is still symmetric with respect to the \( y \)-axis.

Example: Determine if the function \( f(x) = x^3 + x^2 \) is odd or even.
Let’s replace \( x \) with \(-x\) and see what happens. \( f(-x) = (-x)^3 + (-x)^2 = -x^3 + x^2 \). Is this equal to \( f(x) \) or \(-f(x)\)? Well, \( f(x) \) is written above, and \(-f(x) = -x^3 - x^2 \), which is neither of these things. Thus \( f(x) \) is neither odd nor even.

There is a nice trick with polynomials that are odd and even. If a polynomial is odd, all exponents in the polynomial will be odd. If a polynomial is even, all the exponents in the
polynomial will be even. Unfortunately, this only works for polynomials and won’t work with anything else like square roots, rational functions, etc.

Example: Determine if the function \( f(x) = x^5 + x^3 + x \) is odd or even.
Since all the exponents are odd, the function is odd. It is symmetric with respect to the origin.

Example: Determine if \( y = x^4 - 5x^2 \) is odd or even.
Since all the exponents are even, the function is even. It is symmetric with respect to the y-axis.

Example: Determine if \( y = x^{14} - 15x^4 + 2 \) is odd or even.
All the exponents are even, since \( 2 = 2x^0 \), so this graph is even and symmetric about the y-axis.

Example: Determine if the function \( f(x) = x^3 + x + 1 \) is odd or even.
We have two odd exponents and one even exponent (the 1 is \( 1x^0 \)), so this function is neither even nor odd.
Greatest Integer Function

\[ f(x) = \lfloor x \rfloor \]  
(also called a step function)

→ This function pairs every real number \( x \) with the greatest integer less than or equal to \( x \).

The First Step

<table>
<thead>
<tr>
<th>( x )</th>
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<tbody>
<tr>
<td>0</td>
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</tbody>
</table>

In general, if \( 0 \leq x < 1 \) then \( y = 0 \)

The Next Step

<table>
<thead>
<tr>
<th>( x )</th>
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<tbody>
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<td>1.999</td>
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</table>

In general, if \( 1 \leq x < 2 \) then \( y = 1 \)
A Negative Step

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
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</thead>
<tbody>
<tr>
<td>$-1$</td>
<td>$-1$</td>
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<tr>
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<td>$-1$</td>
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<tr>
<td>$0$</td>
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</table>

In general, if $-1 \leq x < 0$ then $y = -1$

The complete graph continues

Domain: $(-\infty, \infty)$

Range: $\{...-2,-1,0,1,2,...\}$
We can use the general pattern established above to graph this new function.

Greatest Integer Function $f(x) = \lfloor 0.5x + 1 \rfloor$:  
- $-1 \leq x < 0 \quad \rightarrow \quad y = -1$
- $0 \leq x < 1 \quad \rightarrow \quad y = 0$
- $1 \leq x < 2 \quad \rightarrow \quad y = 1$ ...

For this new function we have,

- $-1 \leq 0.5x + 1 < 0 \quad \rightarrow \quad y = -1$
- $0 \leq 0.5x + 1 < 1 \quad \rightarrow \quad y = 0$
- $1 \leq 0.5x + 1 < 2 \quad \rightarrow \quad y = 1$ ...

This is based on the greatest integer symbol being applied to the $0.5x + 1$.

To make use of this information and graph the function we need to work out what this means in terms of just $x$ (rather than $0.5x + 1$). Take each inequality and solve for $x$.

- $-1 \leq 0.5x + 1 < 0$  
  $-2 \leq 0.5x < -1$  
  $-4 \leq x < -2$

- $0 \leq 0.5x + 1 < 1$  
  $-1 \leq 0.5x < 0$  
  $-2 \leq x < 0$

- $1 \leq 0.5x + 1 < 2$  
  $0 \leq 0.5x < 1$  
  $0 \leq x < 2$

This will create 3 steps of the graph. This will be enough to observe the pattern and then draw the complete graph.
Example #3 on page 153  

Graph $f(x) = [[0.5x + 1]]$

We can use the general pattern established above to graph this new function.

Notice that the graph has steps that are lengthened due to the 0.5 multiplier. The graph is also shifted to the left by 1 unit.
Week 7

Test Scheduling and Taking the Test

Scheduling a Testing Appointment

- In order to take a test, you must schedule a reservation time.
- Without a reservation, you will not be admitted to the testing room or allowed to take a make-up exam.
- Please recognize that unless you receive a confirmation number and/or confirmation email, you are not registered for your test!
- Registration closes before the first day of testing. Test scheduling open and close dates are listed online in the test scheduling environment.

If you fail to schedule a test by the deadline, you will receive a zero for that exam. The final exam is the only exception to this policy.

To Make a Reservation for a Testing Session

- Log in to MyLabsPlus through the website www.ucf.mylabsplus.com
- Click on your course.
- Click the "Test Scheduling" link on the left-hand menu bar.
- Enter your NID and last name (first letter capitalized).
- Once you've successfully logged into the reservation system, click on a date to create a reservation. The testing dates for each test are listed in the syllabus.
- After deciding on the best available date and time, confirm your email address and complete your reservation.
- Confirm your reservation by checking your Knights email account for the confirmation email.
- You may log into the test scheduling system to confirm your testing appointment. Provided test scheduling is still open, you can also change your reservation.
- Please be aware that there are select dates when the test scheduling will be open to students. These dates will be announced and are posted on the test scheduling website.

Test Taking

To be admitted to the testing session, you must have three things:

1. A testing reservation
2. Your UCF ID (no other ID will be accepted)
3. A new 8.5"x11" Blue Book (smaller Blue Books are unacceptable)

It is also highly recommended that you bring the following as well

- Pen or pencil
- TI-30XA calculator (no other calculator is permitted)
- Knowledge of your MyLabsPlus login and password

Textbook Section 2.8
Function Operations and Composition

Objectives

- The student will be able to calculate using the sum, difference, product, quotient, and composition of functions
- The student will be able to create a new function using the sum, difference, product, quotient, and composition
- The student will be able to find the domain of a function involving the sum, difference, product, quotient, and composition
- The student will be able to calculate the difference quotient for a given function
- The student will be able to solve applications

Key Concepts

Sum:

\[(f + g)(x) = f(x) + g(x)\]

Difference:

\[(f - g)(x) = f(x) - g(x)\]

Product:

\[(fg)(x) = f(x) \cdot g(x)\]
Quotient:
\[
\left( \frac{f}{g} \right)(x) = \frac{f(x)}{g(x)} \quad g(x) \neq 0
\]
- The domains of \( f + g \), \( f - g \), and \( f/g \) include all real numbers in the intersection of the domains of \( f \) and \( g \).
- The domain of \( f/g \) includes those real numbers in the intersection of the domains of \( f \) and \( g \) for which \( g(x) \neq 0 \).

The Difference Quotient:
\[
DQ = \frac{f(x + h) - f(x)}{h}, \quad h \neq 0
\]

Composition of Functions:
\[
(p \circ f)(x) = p(f(x))
\]
The domain of \( g \circ f \) is in the domain of \( f \) such that \( f(x) \) is in the domain of \( g \).

Textbook Section 3.1
Quadratic Functions and Models

Objectives

- The student will be able to graph a quadratic function
- The student will be able to find the vertex, axis, domain, range of a quadratic function
- The student will be able to find the \( x \) and \( y \) - intercepts of a quadratic function
- The student will be able to solve applied problems with quadratic functions

Key Concepts

Quadratic function:
Standard form \( f(x) = ax^2 + bx + c \)
Vertex Form \( f(x) = a(x - h)^2 + k \)
Basic Shape: parabola
Vertex: point at the tip of the parabola, \( (h, k) \)
\[
h = \frac{-b}{2a} \quad k = f(k)
\]
Axis of Symmetry: \( x = h \)
The Graph:
- \( h \): determines a left or right translation
- \( k \): determines an up or down translation
- \( a \): determines vertical stretch/shrink

Domain = \( (-\infty, \infty) \)
Range = \( [k, \infty) \) if \( a > 0 \) (parabola opens up)
Range is \( (-\infty, k] \) if \( a < 0 \) (parabola opens down)

When solving for \( x \)-intercepts, there are 3 possible scenarios:
- Two real solutions: Two \( x \)-intercepts
- Complex solution: No \( x \)-intercepts
- One real solution: One \( x \)-intercept
Textbook Section 3.2
Synthetic Division

Objectives

- The student will be able to use synthetic division to divide
- The student will be able to express \( f(x) \) in the form \( f(x) = (x - k) q(x) + r \) for given \( k \)
- The student will be able to use the remainder theorem and synthetic division to find \( f(k) \)
- The student will be able to determine whether a given value of \( k \) is a zero of a polynomial

Key Concepts

Division Algorithm:

\[
\frac{f(x)}{(x - a)} = q(x) + \frac{r(x)}{(x - a)}
\]

Special Case:

\[
f(x) = (x - k)q(x) + r
\]

Remainder Theorem:

If the polynomial \( f(x) \) is divided by \( x - k \), then the remainder is equal to \( f(k) \).

A zero of a polynomial function \( f \) is a number \( k \) so that \( f(k) = 0 \). Real number zeros are the x-intercepts of the graph.

Handouts

Some of the files you are about to view/download are PDF files. If you do not have Adobe Acrobat installed on your system, you can download the free Adobe Acrobat Reader at [http://www.adobe.com/products/acrobat/readstep.html](http://www.adobe.com/products/acrobat/readstep.html)

- Zeros of Polynomial Functions Handout

Online Homework and Quiz Assignments

After reviewing the Key Concepts and Handouts, log into MyLabPlus and begin your homework and quiz for this week, go to [www.ucf.mylabsplus.com](http://www.ucf.mylabsplus.com) and begin working on your assignments.

Reminders

This week, you will be scheduling a testing appointment. Please be sure to confirm that you have an appointment by clicking on check reservation when you have completed the process. There are several helpful handouts for the course material this week. Don't forget to complete the practice tests.
Function Operations and Composition, 2.8

There are ways we can combine functions to produce new functions. We will consider five. Let \( f(x) \) and \( g(x) \) be functions. For all values of \( x \) for which both \( f(x) \) and \( g(x) \) are defined, we have the following functions:

- **Sum**: \((f + g)(x) = f(x) + g(x)\)
- **Difference**: \((f - g)(x) = f(x) - g(x)\)
- **Product**: \((f \cdot g)(x) = f(x) \cdot g(x)\)
- **Quotient**: \(\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}\)

**Domain:**
- The domains of \( f + g, f - g, \) and \( f \cdot g \) include all real numbers in the intersection of the domain of \( f \) and the domain of \( g \).
- The domain of \( \frac{f}{g} \) includes those real numbers in the intersection of the domain of \( f \) and the domain of \( g \) for which \( g(x) \neq 0 \).

**Example: #4**

Let \( f(x) = x^2 + 3 \) and \( g(x) = -2x + 6 \). Find \( (fg)(4) \).

**Solution:**
We calculate

\[
\begin{align*}
(fg)(4) &= f(4) \cdot g(4) \\
(fg)(4) &= f(4^2 + 1)(-2(4) + 6) \\
(fg)(4) &= (19)(-2) \\
(fg)(4) &= -38
\end{align*}
\]

**Example: #28**

Two functions are given below in a table. Evaluate \( \left(\frac{f}{g}\right)(0) \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( g(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-4</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>8</td>
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<td>4</td>
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<tr>
<td>4</td>
<td>0</td>
<td>0</td>
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</tbody>
</table>

**Solution:** We look up \( f(0) = 8 \) and \( g(0) = -1 \) in the table and calculate

\[
\left(\frac{f}{g}\right)(0) = \frac{f(0)}{g(0)} = \frac{8}{-1} = -8
\]

**Example: #12**

Let \( f(x) = 4x^2 + 2x, g(x) = x^2 - 3x + 2 \)

Find \( f + g, fg \), and \( \frac{f}{g} \). Give the domain of each.

**Solution:**

- \( D_f = (-\infty, \infty) \)
- \( D_g = (-\infty, \infty) \)
- \( f + g = (4x^2 + 2x) + (x^2 - 3x + 2) = 5x^2 - x + 2 \)
- \( fg = (4x^2 + 2x)(x^2 - 3x + 2) \)
- \( \frac{f}{g} = \frac{4x^2 + 2x}{x^2 - 3x + 2} \)
b. \( f_g = (4x^2 + 2x)(x^2 - 3x + 2) \)
\[ - 4x^4 - 12x^3 + 8x^2 + 2x^2 - 6x^2 + 4x \]
\[ = -4x^4 - 10x^3 + 2x^2 + 4x \]
\( D_{fg} = (-\infty, \infty) \)

c. \( f + g = \frac{4x^2 - 2x}{x^2 - 2x + 2} \)
\[ \text{no need to reduce.} \]

Domain:
Exclude values that make the denominator, \( g(x) \), zero.
\[ x^2 - 3x + 2 = 0 \]
\[ (x-1)(x-2) = 0 \]
\( x = 1 \) \( x = 2 \) → Exclude these values
\[ D_{f + g} = (-\infty, 1) \cup (1, 2) \cup (2, \infty) \]

For the pair of functions defined, find \( f + g, f - g, fg \), and \( \frac{f}{g} \). Give the domain of each.

**#14** \( f(x) = \sqrt{5x - 4}, g(x) = -\frac{3}{x} \)

Domain of \( f \) → set the argument to be greater than or equal to 0
\[ 5x - 4 \geq 0 \]
\[ x \geq \frac{4}{5} \]
\[ D_f = \left[ \frac{4}{5}, \infty \right) \]

Domain of \( g \) → exclude values that make the denominator 0
\[ x = 0 \]
\[ D_g = (-\infty, 0) \cup (0, \infty) \]

a. \( f + g = \sqrt{5x - 4} + \left( -\frac{3}{x} \right) \)
\[ = \frac{\sqrt{5x - 4} - 3}{x} \]

The domain is the intersection of \( D_f \) and \( D_g \)

(Draw a number line picture of each and where they overlap.)
\[ D_{f + g} = \left( \frac{4}{5}, \infty \right) \]

b. \( f - g = \sqrt{5x - 4} - \left( -\frac{3}{x} \right) \)
\[ = \frac{\sqrt{5x - 4} + 3}{x} \]

The domain is the intersection of \( D_f \) and \( D_g \)
\[ D_{f - g} = \left[ \frac{4}{5}, \infty \right) \]

c. \( fg = \sqrt{5x - 4} \cdot \left( -\frac{3}{x} \right) \)
\[ = -\frac{3\sqrt{5x - 4}}{x} \]

The domain is the intersection of \( D_f \) and \( D_g \)
\[ D_{fg} = \left[ \frac{4}{5}, \infty \right) \]

d. \( \frac{f}{g} = \frac{\sqrt{5x - 4}}{-\frac{3}{x}} \)
\[ = -\frac{x\sqrt{5x - 4}}{-3} \]
The domain is the intersection of \(D_y\) and \(D_f\), excluding the values that make \(g(x) = 0\)

Note here that \(g(x) = -\frac{1}{x}\) can never be 0

\[
D_f = \left[\mathbb{R} \setminus \{0\}\right]
\]

**The Difference Quotient:**

- calculates the slope of the secant line (line touching the graph at 2 points)
- \(h\) represents the distance between these 2 points
- as \(h\) becomes very small (and the 2 points grow closer together) calculates the slope at a single point, this is the derivative

\[
m = \frac{f(x+h) - f(x)}{(x+h) - x}
\]

The difference quotient is

\[
DQ = \frac{f(x+h) - f(x)}{h}, h \neq 0
\]

**Example 3.6**

For the function defined as follows, find

a. \(f(x+h)\)

b. \(f(x+h) - f(x)\)

c. \(\frac{f(x+h) - f(x)}{h}\)

\[
f(x) = 4x + 11
\]

a. \(f(x+h) = 4(x+h) + 11 = 4x + 4h + 11\)

b. \(f(x+h) - f(x) = 4x + 4h + 11 - (4x + 11) = 4h\)

c. \(\frac{f(x+h) - f(x)}{h} = \frac{4h}{h} = 4\)

**Composition of Functions:**

Let \(f(x)\) and \(g(x)\) be functions.

The **composite function** of \(g\) and \(f\) is defined by:

\[(g \circ f)(x) = g(f(x))\]

(Reads ‘\(g\circ f\) of \(x\).’)

Apply 2 functions in succession

Use the output of function \(f\) as the input for function \(g\):

\[x \rightarrow f(x) \rightarrow [g] \rightarrow g(f(x))\]

**Domain:**

The domain of \(g \circ f\) is the set of all numbers \(x\) in the domain of \(f\) such that \(f(x)\) is in the domain of \(g\).

\(x\)-values in the domain satisfy 2 criteria:
1. \( x \) is the input into \( f(x) \), so exclude \( x \)-values that make \( f(x) \) undefined.

2. \( k(x) \) is the input into \( g(k(x)) \), so exclude values of \( f(k(x)) \) that make \( g(x) \) undefined (then solve to find excluded \( x \)-values).

Let \( f(x) = 2x - 3 \), and \( g(x) = -x + 3 \).

Find each function value:

**Example \#46**

\[
\begin{align*}
\quad \text{Scratch Work} \\
(g \circ f)(-2) &= g(f(-2)) \\
&= g(-7) \\
&= -(-7) + 3 \\
&= 10
\end{align*}
\]

**Example \#43**

\[
\begin{align*}
\text{Scratch Work} \\
(f \circ g)(-2) &= f(g(-2)) \\
&= f(5) \\
&= 2(5) - 3 \\
&= 7
\end{align*}
\]

**NOTE** \((g \circ f)(-2) = 10 \) and \((f \circ g)(-2) = 7\)

In general, \((g \circ f)(x) = (f \circ g)(x)\)

Exception: when \( g \) and \( f \) are inverses of each other (ch 4)

Given functions \( f \) and \( g \), find

a. \((f \circ g)(x)\) and its domain, and

b. \((g \circ f)(x)\) and its domain

**Example \#60**

\[
\begin{align*}
f(x) &= \sqrt{x}, \quad g(x) = x - 1 \\
D_f &= [0, \infty) \\
D_g &= (-\infty, \infty)
\end{align*}
\]

a. \((f \circ g)(x) = f(g(x))\)

\[
\begin{align*}
&= f(p-1) \\
&= \sqrt{p-1} \\
&= \sqrt{\sqrt{\sqrt{x}}}
\end{align*}
\]

\( x \to [g] \to g(x) \to [f] \to g(f(x)) \)

Domain:

- Exclude those values of \( x \) that make \( g \) undefined
  - None in this case

- Exclude those values of \( g(x) \) that make \( f \) undefined
  \[ g(x) = 0; \quad x - 1 \geq 0; \quad x \geq 1 \]

  \( D_{f \circ g} = [1, \infty) \)

b. \((g \circ f)(x) = g(f(x))\)

\[
\begin{align*}
&= g(\sqrt{x}) \\
&= \sqrt{\sqrt{x}} - 1 \\
&= \sqrt[4]{x} - 1
\end{align*}
\]

\( x \to [f] \to f(x) \to [g] \to g(f(x)) \)
Domain:
- Exclude those values of $x$ that make $f$ undefined
  
  $x \geq 0$

- Exclude those values of $g(x)$ that make $g$ undefined
  No values

$D_{g \circ f} = [0, \infty)$

**Example #71**

$$f(x) = \frac{1}{x^2 - 2}, \ g(x) = \frac{1}{x}$$

$D_f = (-\infty, 2) \cup (2, \infty)$

$D_g = (-\infty, 0) \cup (0, \infty)$

a. $(g \circ f)(x) = g(f(x))$

$$= g\left(\frac{1}{x^2 - 2}\right)$$

$$= \frac{1}{\left(\frac{1}{x^2} - 2\right)}$$

$$= \frac{x^2}{1 - 2x}$$

$x \rightarrow [g] \rightarrow g(x) \rightarrow [f] \rightarrow f(g(x))$

Domain:
Exclude those values of $x$ that make $g$ undefined:

$x \neq 0$

Exclude those values of $g(x)$ that make $f$ undefined:

$g(x) \neq 2$

$$\frac{1}{x} \neq 2$$

$$1 \neq 2x$$

$$x \neq \frac{1}{2}$$

$D_{f \circ g} = (-\infty, 0) \cup \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$

b. $(g \circ f)(x) = g(f(x))$

$$= g\left(\frac{1}{x^2 - 2}\right)$$

$$= \frac{1}{\left(\frac{1}{x^2 - 2}\right)}$$

$$= x - 2$$

$x \rightarrow [f] \rightarrow f(x) \rightarrow [g] \rightarrow g(f(x))$
\[ x = 2 \]

- Exclude those values of \( f(x) \) that make \( g \) undefined

\[ f(x) \neq 0 \]

\[ \frac{1}{x-2} \neq 0 \]

There isn't an \( x \) value that would make this happen.

\[ D_{f \circ g} = [-\infty, 2) \cup (2, \infty] \]

**Decomposition of Functions**

Given \( h(x) \), find functions \( f \) and \( g \) such that \( (f \circ g)(x) = h(x) \)

**Example #81**

\[ h(x) = (6x - 2)^2 \]

\[ g(x) = 6x - 2 \]

\[ f(x) = x^2 \]

**Example #86**

\[ h(x) = \sqrt{2x + 3} - 4 \]

\[ g(x) = 2x + 3 \]

\[ f(x) = \sqrt{x} - 4 \]

**Example #91**

An oil well off the Gulf Coast is leaking, with the leak spreading oil over the water’s surface as a circle. At any time \( t \), in minutes, after the beginning of the leak, the radius of the circular oil slick on the surface is \( r(t) = 4t \) feet. Let \( A(r) = \pi r^2 \) represent the area of a circle of radius \( r \).

a. Find \( (A \circ r)(t) \)

\[ (A \circ r)(t) = \pi(4t)^2 \]

\[ = 16\pi t^2 \]

b. Interpret \( (A \circ r)(t) \)

\( (A \circ r)(t) \) represents the area of the circle at time \( t \).

c. What is the area of the oil slick after 3 minutes?

\[ (A \circ r)(3) = 16\pi(3)^2 \]

\[ = 144\pi \]
Quadratic Functions and Models, 3.1

In general, …

Definition: A polynomial function of degree n, is a function defined by an expression of the form

\[ f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \]

where \( a_n, a_{n-1}, \ldots, a_1, \) and \( a_0 \) are real numbers and \( a_n \neq 0 \).

and where \( n \) is a nonnegative integer.

Recall:

- Degree = highest power on the variable \( (x) \)
  
  (NOTE: This is true when we have only one variable.)

- \( a_0 \) = leading coefficient

More specifically, …

Definition: A function \( f \) is a quadratic function if

\[ f(x) = ax^2 + bx + c \]

where \( a, b, \) and \( c \) are real numbers, with \( a \neq 0 \).

NOTICE: Degree = 2.

Recall: \( y = x^2 \) Basic Squaring Function

![Graph of a parabola with vertex at (0,0)](image)

Standard Form: \( y = ax^2 + bx + c \)

Basic Shape: parabola

Vertex: point at the tip of the parabola \((0,0)\)

Axis of Symmetry: LINE that divides the parabola into two equal halves \( x = 0 \)

\( a > 0: \) parabola opens up

\( a < 0: \) parabola opens down

By completing the square, any quadratic function in standard form can also be written in vertex form:

Vertex Form: \( f(x) = a(x-h)^2 + k \)

Graphing Techniques

\( h \) determines a left or right translation (shift)

\( k \) determines a up or down translation (shift)

\( a \) determines vertical stretching or shrinking

\( (NOTE: \) This is the same \( a \) as in standard form.)

Vertex = \((h, k)\) (NOTE: Shifting the graph forms the vertex)

Axis: \( x = h \)
Domain and Range

For the quadratic function defined by \( f(x) = a(x - h)^2 + k \)

- Domain = \((-\infty, \infty)\)
- If \( a > 0 \), then the range is \([k, \infty)\) (parabola opens up)
- If \( a < 0 \), then the range is \((-\infty, k]\) (parabola opens down)

\[ f(x) = (x - 5)^2 - 4 \]

Given the equation and the graph of a quadratic function, find the following:

\[
\begin{align*}
\text{a.} & \quad \text{Give the domain and range.} \\
\text{Domain} & = (-\infty, \infty) \\
\text{Range} & = [-4, \infty) \\
\text{b.} & \quad \text{Give the coordinates of the vertex.} \\
\text{Vertex} & = (5, -4) \\
\text{c.} & \quad \text{Give the equation of the axis of symmetry.} \\
\text{x} & = 5 \\
\text{d.} & \quad \text{Find the y-intercept. (Solve or look at graph.)} \\
\text{Set} \; x & = 0 \\
\text{y} & = (0-5)^2 - 4 \\
\text{y} & = 25 - 4 \\
\text{y} & = 21 \\
\text{Point} & = (0, 21) \\
\text{e.} & \quad \text{Find the x-intercept(s). (Solve or look at graph.)} \\
\text{Set} \; y & = 0 \\
0 & = (x-5)^2 - 4 \\
4 & = (x-5)^2 \\
\pm 2 & = x - 5 \\
x & = 5 \pm 2 \\
x & = 7, 3 \\
\text{Point} & = (7, 0) \text{ and } (3, 0)
#17) \( f(x) = \frac{1}{2} (x + 1)^2 - 3 \)

Graph the quadratic equation.

Give the vertex, axis, domain, and range.

\[
\begin{align*}
\sigma &= \frac{-1}{2} \\
h &= -1 \\
k &= -3 \\
\text{Vertex} &= (-1, -3) \\
\text{Axis} &= x = -1
\end{align*}
\]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>-5</td>
</tr>
<tr>
<td>-2</td>
<td>-\frac{7}{2}</td>
</tr>
<tr>
<td>-1</td>
<td>-3</td>
</tr>
<tr>
<td>0</td>
<td>-\frac{7}{2}</td>
</tr>
<tr>
<td>1</td>
<td>-5</td>
</tr>
</tbody>
</table>

Domain = \( (-\infty, \infty) \)

Range = \( (-\infty, -3] \)

#22) \( f(x) = 2x^2 - 4x + 5 \)

Use completing the square to write the equation in vertex form.

Graph the quadratic equation.

Give the vertex, axis, domain, and range.

\[
\begin{align*}
y &= 2x^2 - 4x + 5 \\
y &= 2(x^2 - 2x) + 5 \\
y &= 2(x^2 - 2x + 1) + 5 - 2 \\
y &= 2(x - 1)^2 + 3 \\
y &= 2(x - 1)^2 + 3
\end{align*}
\]

\( a = 2 \) opens up

\( h = 1 \)

\( k = 3 \)

\( \text{Vertex} = (1, 3) \)

\( \text{Axis} = x = 1 \)
\begin{tabular}{|c|c|}
\hline
\textbf{x} & \textbf{y} \\
\hline
-1 & 11 \\
0 & 5 \\
1 & 3 \\
2 & 5 \\
3 & 11 \\
\hline
\end{tabular}

Domain = \(\{ -\infty, \infty \} \)

Range = \(\{3, \infty\} \)

\( f(x) = 2x^2 - 4x + 5 \)

- Find the y-intercept. Find \( f(0) \).

\[ f(0) = 2(0)^2 - 4(0) + 5 \]

\[ f(0) = 5 \]

Point = \((0, 5)\)

- Find the x-intercept. Let \( y = 0 \). Solve for \( x \).

\[ 0 = 2x^2 - 4x + 5 \]

\[ x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(5)}}{2(2)} \]

\[ x = \frac{4 \pm \sqrt{16 - 40}}{4} \]

\[ x = \frac{4 \pm \sqrt{-24}}{4} \]

\[ x = 1 \pm \frac{\sqrt{6}}{2} \]

Not "plot-able". No x-intercepts.

**In general: When solving for x-intercepts, (pg. 247)**

- If \( b^2 - 4ac > 0 \), the x-intercepts are \( \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \).
- If \( b^2 - 4ac = 0 \), the x-intercept is \( -\frac{b}{2a} \).
- If \( b^2 - 4ac < 0 \), there are no x-intercepts.

Match the following criteria to the appropriate graph:

#31) \( a < 0 \); \( b^2 - 4ac = 0 \) opens down, one x-int.

#32) \( a > 0 \); \( b^2 - 4ac < 0 \) opens up, no x-int.

#33) \( a < 0 \); \( b^2 - 4ac < 0 \) opens down, no x-int.
The Vertex Formula

The quadratic function defined by \( f(x) = ax^2 + bx + c \) has the following formula to find the vertex \((h, k)\):

\[
h = -\frac{b}{2a}, \quad k = f(h) \quad \text{(plug in } h \text{ to find } k)\]

\#26) \( f(s) = \frac{2}{3}s^2 - \frac{8}{3}s + \frac{5}{3} \)

Graph the quadratic equation.

Given the vertex, axis, domain, and range:

\[
a = \frac{2}{3}, b = -\frac{8}{3}, c = \frac{5}{3}
\]

\[
h = -\frac{-\frac{8}{3}}{2 \cdot \frac{2}{3}} = -\frac{2}{3} = \frac{1}{3}
\]

\[
k = f(h) = \frac{2}{3}(\frac{1}{3})^2 - \frac{8}{3}(\frac{1}{3}) + \frac{5}{3} = \frac{2}{3} - \frac{8}{9} + \frac{5}{3} = \frac{-3}{3} = -1
\]

If \( h = 2 \), then \( k = f(2) \):

\[
k = f(2) = \frac{2}{3}(2)^2 - \frac{8}{3}(2) + \frac{5}{3} = \frac{8}{3} - \frac{16}{3} + \frac{5}{3} = \frac{-3}{3} = -1
\]

Vertex = \((2, -1)\)

Axis: \( x = 2 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5/3</td>
</tr>
<tr>
<td>1</td>
<td>1/3</td>
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<tr>
<td>2</td>
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<td>3</td>
<td>1/3</td>
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<tr>
<td>4</td>
<td>5/3</td>
</tr>
</tbody>
</table>

Domain = \([-\infty, \infty]\)

Range = \([-1, \infty]\)
Quadratic Models:

Opens up \((a > 0)\)
Vertex = minimum

Opens down \((a < 0)\)
Vertex = maximum

49. **Minimum Cost**

Ms. Harris has a taco stand. She has found that her daily costs are approximated by:

\[ C(x) = x^2 - 40x + 610 \]

where \(C(x)\) is the cost, in dollars, to sell \(x\) units of tacos.

Find the number of units of tacos she should sell to minimize her costs. What is the minimum cost?

\(a\) is positive \(\rightarrow\) Parabola opens up \(\rightarrow\) Vertex is a minimum.

\[ x = -\frac{b}{2a} \]

\[ = -\frac{-40}{2(1)} \]

\[ = 20 \]

Ms. Harris should sell 20 units of tacos to minimize her costs

\[ C(20) = (20)^2 - 40(20) + 610 \]

\[ C(20) = 210 \]

Ms. Harris’s minimum cost is $210

---

52. **Projectile Motion: (If air resistance is neglected)**

The height \(s\) (in feet) of an object projected directly upward from an initial height \(s_0\) feet with initial velocity \(v_0\) feet per second is:

\[ s(t) = -16t^2 + v_0t + s_0 \]

53. **Height of a Toy Rocket**

A toy rocket is launched straight up from the top of a building 50 ft tall at an initial velocity of 200 ft per sec.

a. Give the function that describes the height of the rocket in terms of time \(t\):

\[ s(t) = -16t^2 + 200t + 50 \]

b. Determine the time at which the rocket reaches its maximum height, and the maximum height in feet.

Parabola opens down: vertex = max

\[ x = -\frac{200}{2(-16)} \]

\[ = \frac{200}{32} \]

\[ = 6.25 \]

The rocket will reach its maximum height in 6.25 seconds

\[ s(6.25) = -16(6.25)^2 + 200(6.25) + 50 \]

\[ = 675 \]
The rocket's maximum height is 675 feet.

c. For what time interval will the rocket be more than 300 ft above ground level?

\[
300 < -16t^2 + 200t + 50
\]

\[
0 < -16t^2 + 200t - 250
\]

\[
0 = -16t^2 + 200t - 250
\]

\[
t = \frac{-200 \pm \sqrt{200^2 - 4(-16)(-250)}}{2(-16)}
\]

\[
t = \frac{-200 \pm 2000}{-32}
\]

\[
t = \frac{200 \pm 40\sqrt{15}}{32}
\]

\[
t = 11.091, t = 1.009
\]

Could test points with the chart learned in 17, or.....

Let's look at what's going on with a picture.

![Graph of rocket's height vs time](image)

From the graph, we see the solution is (1.009, 11.091)

d. After how many seconds will it hit the ground?

"Hit the ground" → height = 0 → \( a = 0 \)

*Note: \( a(t) \) represents height \( a \) written as a function.

Set \( a(t) = -16t^2 + 200t + 50 \) equal to zero and solve.

\[
0 = -16t^2 + 200t + 50
\]

\[
t = \frac{-200 \pm \sqrt{200^2 - 4(-16)(50)}}{2(-16)}
\]

\[
t = \frac{-200 \pm \sqrt{43200}}{-32}
\]

\[
t = \frac{200 \pm 120\sqrt{3}}{32}
\]

\[
t = 12.745, t = -0.245
\]

The rocket will hit the ground in 12.745 seconds.
#57) Volume of a Box

A piece of cardboard is twice as long as it is wide. It is to be made into a box with an open top by cutting 2-in squares from each corner and folding up the sides. Let \( x \) represent the width (in inches) of the original piece of cardboard.

Sketch a picture.

\[ \begin{array}{c}
| & 1 & 2 & | \\
<table>
<thead>
<tr>
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\end{array} \]

a. Represent the length of the original piece of cardboard in terms of \( x \).
   \[ \text{length} = 2x \]

b. What will be the dimensions of the bottom rectangular base of the box? Give the restrictions on \( x \).
   \[ \text{width} = x - 4 \]
   \[ \text{length} = 2x - 4 \]
   \[ \text{Restrictions on } x: x < 4 \]
   Need a positive measurement for the width.

c. Determine a function \( V \) that represents the volume of the box in terms of \( x \).
   \[ V = l \cdot w \cdot h \]
   \[ V(x) = (2x - 4)(x - 4)(2) \]
   \[ V(x) = (2x^2 - 12x + 16)(2) \]
   \[ V(x) = 4x^2 - 24x + 32 \]

d. For what dimensions of the bottom of the box will the volume be 320 in\(^3\)?
   \[ 320 = 4x^2 - 24x + 32 \]
   Divide by 4
   \[ 0 = 4x^2 - 24x + 32 \]
   Factor or use quadratic formula
   \[ 0 = (x - 4)(x - 8) \]
   \[ x = 4, x = 8 \]
   Is it reasonable for \( x \) to be -8? Look back at part b).

Width of box: \( x - 4 = 4 - 4 = 0 \)
Length of box: \( 2x - 4 = 2(4) - 4 = 4 \)

The width of the box is 8 in. The length of the box is 20 in.
Synthetic Division, 3.2

Division Algorithm:
Let \( f(x) \) and \( g(x) \) be polynomials with \( g(x) \) of lower degree than \( f(x) \) and \( g(x) \) of degree one or more.

There exist unique polynomials \( q(x) \) and \( r(x) \) such that
\[
f(x) = g(x) \cdot q(x) + r(x),
\]
where either \( r(x) = 0 \) or the degree of \( r(x) \) is less than the degree of \( g(x) \).

Solve #9 using long division.

\[
\begin{array}{rll}
  & -9x^3 + 8x^2 - 7x + 2 \\
\hline 
  x - 2 & -9x^3 + 8x^2 - 7x + 2 \\
  \hline 
  & -9x^3 + 10x^2 - 27 \\
  \hline 
  & -32 \\
  \hline
\end{array}
\]

(Fill in by hand)

Solution:
\[
\begin{array}{rll}
  & -9x^3 + 8x^2 - 7x + 2 \\
\hline 
  x - 2 & -9x^3 + 8x^2 - 7x + 2 \\
  \hline 
  & -9x^3 + 10x^2 - 27 \\
  \hline 
  & -32 \\
  \hline
\end{array}
\]

Multiply both sides by \( (x-2) \)
\[
-9x^3 + 8x^2 - 7x + 2 = [(-9x^3 + 10x^2 - 27)](x-2) - 32
\]

Looking at the division algorithm, you can see that
\[
f(x) = -9x^3 + 8x^2 - 7x + 2 \\
g(x) = -9x^2 - 10x - 27 \\
q(x) = x - 2 \\
r(x) = -32
\]

Shortcut to long division: Synthetic Division

#9) (using synthetic division) \[
\begin{array}{rll}
  & -9x^3 + 8x^2 - 7x + 2 \\
\hline 
  2 & -9x^3 + 8x^2 - 7x + 2 \\
  \hline 
  & -9x^3 + 10x^2 - 27 \\
  \hline 
  & -32 \\
  \hline
\end{array}
\]

Solution:
\[
-9x^3 + 10x^2 - 27 - 32
\]

#14) \[
\begin{array}{rll}
  & x^4 + 5x^3 - 6a^2 + 2x \\
\hline 
  -1 & x^4 + 5x^3 - 6a^2 + 2x \\
  \hline 
  & -1x^4 + 4x^3 - 10x^2 + 12 \\
  \hline 
  & -12x + 12 \\
  \hline
\end{array}
\]

Solution:
\[
x^3 + 4x^2 - 10x + 12 - 12
\]

#11) \[
\begin{array}{rll}
  & \frac{1}{3}x^4 - \frac{2}{3}x^2 + \frac{1}{2}x + 1 \\
\hline 
  \frac{1}{3} & \frac{1}{3}x^4 - \frac{2}{3}x^2 + \frac{1}{2}x + 1 \\
  \hline 
  & \frac{1}{3}x^3 + \frac{2}{3}x^2 + \frac{1}{6}x - \frac{1}{3} \\
  \hline 
  & \frac{1}{3}
\end{array}
\]
\[
\begin{array}{c|cccc}
1 & 1 & -2 & 1 & 1 \\
\downarrow & 1 & -1 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 \\
\end{array}
\]

Solution:
\[
x^2 + x + 1
\]

\#17)
\[
\frac{x^3 + 1}{x - 1}
\]

-1 \qquad 1 \qquad 0 \qquad 0 \qquad 0 \\
\downarrow \qquad 1 \qquad 1 \qquad 1 \qquad 0 \\
1 \qquad -1 \qquad 1 \qquad 1 \qquad 0 \\
x^4 \quad x^3 \quad x^2 \quad x \quad \text{C} \quad \text{R}

Solution:
\[
x^4 - x^3 + x^2 - x + 1
\]

******

Special Case of the Division Algorithm:

For any polynomial \(f(x)\) and any complex number \(k\), there exists a unique polynomial \(q(x)\) and number \(r\) such that
\[
f(x) = (x - k)q(x) + r
\]
Express \(f(x)\) in the form \(f(x) = (x - k)q(x) + r\)

\#20)
\[
f(x) = 2x^3 + 3x^2 - 16x + 10; \ k = -4
\]

-4 \qquad 2 \qquad 3 \qquad -16 \qquad 10 \\
\downarrow \qquad -8 \qquad 20 \qquad -16 \\
2 \qquad -5 \qquad 4 \qquad -6 \\
f(x) = (x + 4)[2x^2 - 5x + 4] - 6

\#25)
\[
f(x) = 3x^3 + 4x^2 - 10x^2 + 15; \ k = -1
\]

-1 \qquad 3 \qquad 4 \qquad -10 \qquad 0 \qquad 15 \\
\downarrow \qquad -3 \qquad -1 \qquad 11 \qquad 11 \\
3 \qquad 1 \qquad -11 \qquad 11 \qquad 4 \\
f(x) = (x + 1)[3x^2 - 11x + 11] + 4

Remainder Theorem:

If the polynomial \(f(x)\) is divided by \(x - k\),
then the remainder is equal to \(f(k)\)

For each polynomial function, use the remainder theorem and synthetic division to find \(f(k)\)

\#29)
\[
f(x) = 2x^2 - 3x - 3; \ k = 2
\]
(Calculate \(f(2)\) first.)

**NOTE:**
\[
f(2) = 2(2)^2 - 3(2) - 3
f(2) = -1
\]
(We should get -1 as a remainder when using the remainder theorem.)

-2 \qquad 2 \qquad -3 \qquad -3 \\
\downarrow \qquad 4 \qquad 2 \\
2 \qquad 1 \qquad -1 \quad \Rightarrow \quad \text{The remainder is } -1.
So, \( f(2) = -1 \)

For each polynomial function, use the remainder theorem and synthetic division to find \( f(k) \).

#33)

\[
\begin{array}{c|cccc}
3 & 2 & 0 & -10 & -19 & 0 & -50 \\
- & 6 & 18 & 24 & 15 & 45 \\
\hline
2 & 6 & 8 & 5 & 15 & -5 \\
\end{array}
\]

So, \( f(3) = -5 \)

#38)

\[
f(x) = x^3 - x + 3; k = 3 - 2i
\]

\[
\begin{array}{c|c}
3 - 2i & 1 & -1 & 3 \\
\hline
1 & 2 - 2i & 5 - 10i \\
\end{array}
\]

Check:

\[
f(3 - 2i) = (3 - 2i)^3 - (3 - 2i) + 3
\]

\[
= 5 - 12i - 3 + 2i + 3
\]

\[
= 5 - 10i
\]

On scratch paper,

\[
(3 - 2i)(3 - 2i) = 9 - 6i - 6i + 4
\]

A zero of a polynomial function \( f \) is a number \( k \) such that \( f(k) = 0 \). The real number zeros are the \( x \)-intercepts of the graph of the function.

Use synthetic division to decide whether the given number \( k \) is a zero of the given polynomial function.

If it is not, give the value of \( f(k) \)

#49)

\[
f(x) = 2x^4 + 3x^3 - 8x^2 - 2x + 15; k = -\frac{3}{2}
\]

\[
\begin{array}{c|cccc}
-\frac{3}{2} & 2 & 3 & -8 & -2 & 15 \\
\hline
2 & 3 & -8 & 12 & -15 \\
\end{array}
\]

Hence

\[
f\left(-\frac{3}{2}\right) = 0
\]

and \( k \) is a zero of the function.

**NOTE:** #33 and #38 above did NOT display zeros.

#53)

\[
f(x) = x^2 - 2x + 2; k = 1 - i
\]

\[
1 - i & 1 & -2 & 2 \\
\hline
1 & -1 - i & -2 \\
\end{array}
\]

On scratch,

\[
(1 - i)(1 - i) = -1 + i^2
\]

\[
= -2
\]
FINDING ZEROS OF HIGHER ORDER POLYNOMIALS

When looking for zeros in a higher order polynomial, the first step is typically to use the rational zeros theorem in order to provide a starting point by generating a finite list of possible numbers to test using synthetic division. Unfortunately, a finite list of numbers can still be rather large. That is why some other methods may be useful for determining where the most focus should be.

Reducing the Polynomial

Recall that once a zero, \( k \), has been identified, then \( (x - k) \) is a factor of the original polynomial. By using synthetic division, you have also already “reduced” the original polynomial by a factor of \( (x - k) \). In general, it is far easier to find zeros of this “reduced” polynomial than it will be to find the zeros of the original, and any zero of the reduced polynomial will also be a zero of the original. So every time a new factor is found, work with the reduced polynomial to find the next zero/factor. Regenerating a (now reduced) list of possible rational zeros may be helpful as well. This is why our goal is always to factor until all factors are linear (already solved) or quadratic (easily solved using the formula).

Descartes’ Rule of Signs

This method can help one focus attention on positive or negative numbers, although it needs to be pointed out that Descartes’ rule of signs does not account for whether a number is rational. For example, \( 5 \pm \sqrt{2} \) may be the only two zeros of a function, and while both are positive, neither would show up on a list of possible rational zeros because they are irrational. Despite this limitation, here are some useful tips for applying Descartes’ rule of signs to a list of possible rational zeros.

Method 1: “The Sure Thing”

Suppose Descartes’ rule of signs determines there are 4, 2, or 0 positive zeros, and 3 or 1 negative zeros. While one may be tempted to focus on the possible rational zeros that are positive (either because there could be up to four of them or because they are slightly easier to work with), there may in fact be none at all. So attention should shift to the negatives, since we know there is at least one such zero.

Method 2: “Greater potential”

Suppose Descartes’ rule of signs determines there are 3 or 1 positive zeros and 1 negative zero. While Method 1 would not necessarily distinguish between these two choices, there is potential for there to be 3 times as many zeros among the positives as there are in the negatives (and at worst it is the same number). So attention in this case should shift to the positive, since we are more likely to hit upon a zero there.

The continuous nature of polynomial functions allows for additional methods to aid in the search for zeros.
Remainder Theorem

Once we start testing points using synthetic division, the remainder theorem can be very useful in choosing future possibilities for testing. Recall that the remainder theorem states that the remainder from a synthetic division of a polynomial \( p(x) \) by \((x - k)\) is the same value as \( p(k) \).

Method 1: “Are we there yet?”

Because polynomials are continuous smooth curves, a zero is more likely to occur near a value where \(|p(k)|\) is small rather than when \(|p(k)|\) is large. For example, consider a polynomial where \( p(1) = 15 \) and \( p(10) = 1 \). Choosing other rational zero candidates near \( x = 10 \) makes more sense than choosing them near \( x = 1 \) because \( f(10) \) is closer to zero than was \( f(1) \).

Method 2: “Which way should we go?”

Consider an example with the following list of possible rational zeros.

\[
\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, ... \pm 120
\]

Now suppose that after unsuccessfully trying several possible rational zeros, we see that \( p(-2) = 10 \), \( p(-1) = 20 \), \( p(1) = 3 \), \( p(2) = 5 \), and \( p(3) = 9 \). While the preceding rule would seem to indicate \( x = 4 \) should be our next attempted (since \( p(3) \) is closer to zero than \( p(-2) \), notice the direction of the function (increasing or decreasing) that seems to be taking place, as this could indicate the better location to look for a zero. Essentially, it may be more beneficial to look near \( x \) values that have large function values but are headed \textit{toward} the \( x \)-axis than it would be to look near \( x \) values that may be small, but are headed \textit{away} from the \( x \)-axis.

Method 3: Intermediate Value Theorem

The Intermediate Value Theorem, as used for finding polynomial zeros, says that for any polynomial function, if \( f(a) > 0 \) and \( f(b) < 0 \), then there exists a zero between \( a \) and \( b \). In other words, if one function value is positive and one is negative, a zero exists somewhere between them. For example if \( p(-2) = 3 \), \( p(2) = 1 \), and \( p(4) = -5 \), there must be a zero between \( x = 2 \) and \( x = 4 \). Obviously when searching for zeros, focus should be on areas where the function has a change in sign.

Boundedness Theorem

This is perhaps the most useful of the methods, although it only applies if all coefficients of the polynomial are real and the leading coefficient is positive. If, after performing a synthetic division on some factor \((x - k)\) where \( k > 0 \), all of the numbers in the bottom row of the synthetic division are positive, then there will be no zeros greater than \( k \) (so do not bother testing them). Similarly, if \( k < 0 \) and all the numbers in bottom row alternate sign (positive then negative then positive then negative, etc), then there are no zeros less than \( k \).

One Final Note

Please remember that, depending on the function, there may in fact be no rational zeros or even no real zeros at all. In this instance, none of these tips would prove particularly effective. However, the steps outlined here will still aid you in reaching the appropriate conclusion. Additionally, the likelihood of encountering such a problem in this course is slim.
Week 8

Textbook Section 3.3
Zeros of Polynomial Functions

Objectives

- The student will be able to use the factor theorem and synthetic division to determine whether one polynomial is a factor of another polynomial.
- The student will be able to factor \( f(x) \) into linear factors given a zero of \( f(x) \).
- The student will be able to find all (real or complex) zeros of a polynomial and their multiplicities.
- The student will be able to list possible rational zeros of a polynomial.
- The student will be able to use Descartes' rule of signs.

Key Concepts

Polynomial function:

\[
f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0
\]

A factor \((x - k)\) of a polynomial divides evenly. Setting these factors equal to zero yields the zeros.

Fundamental Theorem of Algebra:

Every function defined by a polynomial function of degree one or more has at least one complex zero.

Number of Zeros Theorem:

A polynomial function of degree \(n\) has at most \(n\) distinct zeros. The number of times a zero occurs is called the multiplicity.

Factor Theorem:

The polynomial \(x - k\) is a factor of the polynomial \(f(x)\) if and only if \(f(k) = 0\).

Conjugate Zeros Theorem:

Let \(f(x)\) be a polynomial function having only real coefficients. If \(z = a + bi\) is a zero of \(f(x)\), then \(z = a - bi\) is a zero of \(f(x)\).

Rational Zeros Theorem:

Let \(f(x)\) be a polynomial function with integer coefficients.

Let \(\frac{p}{q}\) be a rational number written in lowest terms.

If \(\frac{p}{q}\) is a zero of \(f(x)\), then \(p\) is a factor of the constant term and \(q\) is a factor of the leading coefficient.

NOTE: This theorem does not guarantee a zero. It only provides possible rational zeros.

Descartes' Rule of Signs:

Let \(f(x)\) be a polynomial function with real coefficients and a nonzero constant term, with terms in descending powers of \(x\):

- The number of positive real zeros of \(f(x)\) either equals the number of variations in sign occurring in the coefficients of \(f(x)\), or is less than the number of variations by a positive even integer.
- The number of negative real zeros of \(f(x)\) either equals the number of variations in sign occurring in the coefficients of \(f(-x)\), or is less than the number of variations by a positive even integer.

NOTE: Real zeros are not necessarily rational.

-----

Textbook Section 3.4
Polynomial Functions

Objectives
Key Concepts

Graphs of the form: for \( n > 0 \)

- Even exponent: \( n \) \rightarrow \text{Shape: parabola-like}
- Odd exponent: \( n \) \rightarrow \text{Shape: cubic-like}
- For large exponents, \( n \) \rightarrow \text{graph flattens near (0, 0)}
  \text{is steeper at ends}

A turning point is a change in graph from increasing to decreasing or decreasing to increasing.

The graph of a polynomial function with degree \( n \):
- Is continuous,
- Has smooth rounded turns,
- Has at most \( n \) real zeros (x-intercepts),
- Has at most \( n - 1 \) turning points,
- Has at least one turning point between each successive pair of x-intercepts

Multiplicity of Zeros:

the number of times a zero occurs

Suppose that \( k \) is a zero of a polynomial function.

Consider the multiplicity of \( k \):
- If multiplicity = one, then graph crosses x-axis at \((k, 0)\).
- If multiplicity = an even number, then graph is tangent to x-axis at \((k, 0)\).
- If multiplicity = an odd number (greater than one), then the graph crosses AND is tangent to x-axis at \((k, 0)\).

End behavior:

Suppose \( ax^n \) is the dominating term of a polynomial function \( f \).

\[
\begin{align*}
\text{positive } a, \text{ even degree} & \quad \text{positive } a, \text{ odd degree} \\
\text{negative } a, \text{ even degree} & \quad \text{negative } a, \text{ odd degree}
\end{align*}
\]

NOTE: If degree is even, then the shape is parabola-like.

NOTE: If degree is odd, then the shape is cubic-like.

To graph a polynomial function:

Step 1: Find the real zeros of \( f \).
Step 2: Find \( f(0) \).
Step 3: Find the end behavior.
  Use the multiplicity of each zero.
  Find test points.

The Factor Theorem: (restated)

If \( a \) is an x-intercept of the graph of \( y = f(x) \), then
- \( a \) is a zero of \( f \),
- \( a \) is a solution of \( f(a) = 0 \), and
• \( x - a \) is a factor of \( f(x) \).

**Intermediate Value Theorem:**

Let \( f(x) \) be a polynomial function with \textbf{only real coefficients}.

Let \( a \) and \( b \) be real numbers.

If the values \( f(a) \) and \( f(b) \) are opposite in sign, then there exists at least one real zero between \( a \) and \( b \).

**Boundedness Theorem:**

Let \( f(x) \) be a polynomial function of degree \( n \geq 1 \)

• with real coefficients and
• with a positive leading coefficient.

Divide \( f(x) \) by \( x - c \) using synthetic division.

• If \( c > 0 \) and all numbers in the bottom row of the synthetic division are nonnegative, \textbf{then} \( f(x) \) \textbf{has no zero greater than} \( c \).
• If \( c < 0 \) and the numbers in the bottom row of the synthetic division alternate in sign (with 0 considered positive or negative, as needed), \textbf{then} \( f(x) \) \textbf{has no zero less than} \( c \).

\[ \text{-----} \]

**Online Homework and Quiz Assignments**

After completing the Key Concepts and Handouts, log into MyLabsPlus and begin your homework and quiz for this week, go to www.ucf.mylabsplus.com and begin working on your assignments.
Zeros of Polynomial Functions, 3.3

Recall from last time:

- A zero of a polynomial \( f(x) \) satisfies \( f(k) = 0 \).
  
  If \( k \) is a real number, then it will be an \( x \)-intercept of the graph.

- A factor of a polynomial \( (x - k) \) divides evenly.
  
  Factoring a polynomial yields the factors.
  
  Setting these factors equal to zero yields the zeros.

Factor Theorem:

The polynomial \( x - k \) is a factor of the polynomial \( f(x) \) if and only if \( f(k) = 0 \).

(Recall from the Remainder Theorem, \( f(k) = \text{remainder} \).

Use the factor theorem and synthetic division to decide whether the second polynomial is a factor of the first.

\[ \begin{align*}
\#5) \quad x^2 - 5x^2 + 3x + 1; \quad x - 1 \\
\quad \text{Yes, } x - 1 \text{ is a factor.}
\end{align*} \]

\[ \begin{array}{c|cccc}
1 & -5 & 3 & 1 \\
\hline
& 1 & -4 & -1 \\
\hline
& 1 & -4 & 0 \\
\end{array} \]

\[ \begin{align*}
\#7) \quad 2x^4 + 5x^3 - 8x^2 + 3x + 1; \quad x + 1 \\
\quad \text{No, } x + 1 \text{ is NOT a factor.}
\end{align*} \]

\[ \begin{array}{c|cccc}
-1 & 2 & 5 & -8 & 3 & 13 \\
\hline
& -2 & -3 & 11 & 14 \\
\hline
& 2 & -3 & 14 & -1 \\
\end{array} \]

Fundamental Theorem of Algebra:

Every function defined by a polynomial of degree 1 or more has at least one complex zero.

(Recall the degree = highest power on \( x \).)

(Recall the set of real numbers is a subset of the complex numbers.)

Number of Zeros Theorem:

A function defined by a polynomial of degree \( n \) has at most \( n \) distinct zeros.

Definition: The number of times a zero occurs is referred to as the multiplicity of the zero.

If a factor is repeated (exponent on the factor is more than 1), then the number of zeros may be less than the degree of the polynomial.

For each polynomial function, find all zeros and their multiplicities.

\[ \#44) \quad f(x) = (x + 1)^2(x - 1)^3(x^2 - 10) \]

Degree = 7

Zeros:

Set \( x = 0 \rightarrow x = -1 \)
Set \( x + 1 = 0 \rightarrow x = 1 \)
Set \( x^2 - 10 = 0 \rightarrow x^2 = 10 \rightarrow x = \pm \sqrt{10} \)

<table>
<thead>
<tr>
<th>Zeros</th>
<th>-1</th>
<th>1</th>
<th>( \sqrt{10} )</th>
<th>(-\sqrt{10})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mult.</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
# 43) \( f(x) = -7x^3 + x \)

Factor first: \( f(x) = x(7x^2 + 1) \)

Degree = 3

Zeros:

Set \( x = 0 \rightarrow x = 0 \)

Set \( 7x^2 + 1 = 0 \rightarrow 7x^2 = -1 \)

\[ x^2 = \frac{1}{7} \]

\[ x = \pm \frac{\sqrt{7}}{7} \]

\[ x = \pm \frac{\sqrt{7}}{7} \]

Multiply for each:

<table>
<thead>
<tr>
<th>Zeros</th>
<th>( \frac{\sqrt{7}}{7} )</th>
<th>( -\frac{\sqrt{7}}{7} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mult.</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

# 219) \( f(x) = 6x^3 + 13x^2 - 14x + 3; k = -3 \)

Factor \( f(x) \) into linear factors given that \( k \) is a zero of \( f(x) \).

*(NOTE: Linear factors contain \( x \) to the first power.)*

\[
\begin{array}{cccc}
-3 & 6 & 13 & -14 & 3 \\
1 & -18 & 15 & -3 & 0 \\
6 & -5 & 1 & 0 & \\
\end{array}
\]

Therefore,

\[
f(x) = (x + 3)(6x^2 - 5x + 1) \\
f(x) = (x + 3)(3x - 1)(2x - 1)
\]

# 227) \( f(x) = x^4 + 2x^3 - 7x^2 - 20x - 12; k = -2 \)

\[
\begin{array}{cccc}
-2 & 1 & 2 & -7 & -20 & -12 \\
1 & -2 & 0 & 14 & 12 & 0 \\
-2 & 1 & 0 & -7 & -6 & 0 \\
1 & -2 & 4 & 6 & 0 & \\
\end{array}
\]

Therefore,

\[
f(x) = (x + 2)(x + 2)(x^2 - 2x - 3) \\
f(x) = (x + 2)(x + 2)(x + 3)(x - 1)
\]

---

**Conjugate Zeros Theorem:**

Let \( f(x) \) be a polynomial function having only real coefficients.

If \( z = a + bi \) is a zero of \( f(x) \), then \( z = a - bi \) is a zero of \( f(x) \), (where \( a \) and \( b \) are real numbers).

# 31) \( f(x) = x^3 - 2x^2 + 17x - 15; 2 - i \)

For each polynomial function, one zero is given.

Find all the others.

\[
\begin{array}{cccc}
2 - i & 1 & -7 & -15 \\
1 & 2 - i & -11 + 3i & 15 & 0 \\
1 & -5 - i & 6 + 3i & 0 & \\
\end{array}
\]
Conjugate Zeros Theorem says what? **2 + i must be a zero.

\[
\begin{array}{ccc}
2 + i & -5 - i & 6 + 3i \\
1 & 2 + i & -6 - 3i \\
1 & -3 & 0 \\
\end{array}
\]

On scratch:

\((2 - i)(-5 - i) = -10 + 3i + i^2\)

\[= -10 + 3i - 1\]

\[= -11 + 3i\]

\((2 - 1)(6 + 3i) = 12 - 3i^2\)

\[= 12 + 3\]

\[= 15\]

Therefore, \(f(z) = [x - (2 - i)][x - (2 + i)][x - 3]\)

Zeros: \(x = [2 - i, 2 + i, 3]\)

**25** \(f(x) = x^3 - 2x^2 + (3-2i)x^2 + (-8 - 5i)x + (3 + 3i); \) i = 1 + i

For each polynomial function, one zero is given.

Find all the others, and factor into linear factors:

\[
\begin{array}{ccc}
1 + i & 2 & 3 - 2i \\
1 & 2 + i & -8 - 5i \\
2 & 5 & -3 - 3i \\
\end{array}
\]

Does the conjugate zeros theorem apply to this problem?

Why or why not? **No since coefficients are complex:**

\(f(x) = [x - (1 + i)][2x^2 + 5x - 3]\)

\[= [x - (1 + i)][(2x - 1)(x + 3)]\)

Set remaining factors equal to zero.

Zeros: \(\{3, \frac{1}{2}, (1 + i)\}\)

**30** \(f(x) = x^3 + 4x^2 - 5x + 1\

\[
\begin{array}{ccc}
1 & 4 & 0 & -5 \\
1 & 5 & 5 & 0 \\
\end{array}
\]

Therefore, \(f(x) = (x-1)(x^2 + 5x + 5)\)

Find zeros: \((x-1)(x^2 + 5x + 5) = 0\)

\(x-1 = 0\)

\(x = 1\)

\(x^2 + 5x + 5 = 0\)

\(x = \frac{-5 \pm \sqrt{5^2 - 4[1][5]}}{2}\)

\(x = \frac{-5 \pm \sqrt{5}}{2}\)
34) \( f(x) = x^4 + 10x^3 + 27x^2 + 10x + 26 \).}

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>10</th>
<th>27</th>
<th>10</th>
<th>26</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>10+i</td>
<td>-1+i</td>
<td>-10-26i</td>
<td>-26</td>
<td></td>
</tr>
<tr>
<td>( i )</td>
<td>10+i</td>
<td>26+i</td>
<td>26</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Does the conjugate zeros apply to this function?
What is the conjugate of \( i \)?

<table>
<thead>
<tr>
<th>( -1 )</th>
<th>1</th>
<th>10+i</th>
<th>26+10i</th>
<th>26</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>10+i</td>
<td>-1+i</td>
<td>-10+26i</td>
<td>-26</td>
</tr>
<tr>
<td>( x^2 + 10x + 26 = 0 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
x = \frac{-10 \pm \sqrt{100 - 100}}{2}
\]

\[
x = \frac{-10 \pm 0}{2}
\]

\[
x = -5
\]

Zeros = \( \{1, -i, -5 \pm i \} \).

49) Zeros of \(-3, 1, \text{ and } 4; f(2) = 30\)

Find a polynomial function of degree 3 with real coefficients that satisfies the given conditions.

\[
f(x) = a(x + 3)(x - 1)(x - 4)
\]

\[
- a(x + 3)(x^2 - 5x + 4)
\]

\[
= a(x^3 - 2x^2 - 7x + 12)
\]

\[
f(2) = a(2^3 - 2(2)^2 - 7(2) + 12)
\]

\[
30 = a(0 - 8 - 22 + 12)
\]

\[
30 = a(-16)
\]

\[
a = -\frac{3}{4}
\]

\[
f(x) = \frac{-3}{4}(x^3 - 2x^2 - 7x + 12)
\]

\[
= -\frac{3}{4}x^3 + \frac{3}{2}x^2 + \frac{21}{2}x - 9
\]

64) -1 and \( 4 - 2i \)

Find a polynomial function of least degree having only real coefficients.
If \( 4 - 2i \) is a zero, then \( 4 + 2i \) is a zero.

\[
f(x) = (x + 1)(x - (4-2i))(x - (4+2i))
\]

\[
= (x + 1)(x^2 - 8x + 20)
\]

\[
= x^3 - 7x^2 + 12x + 20
\]
Rational Zeros Theorem:

Let \( f(x) \) be a polynomial function with integer coefficients.

Let \( \frac{p}{q} \) be a rational number written in lowest terms.

If \( \frac{p}{q} \) is a zero of \( f \), then \( p \) is a factor of the constant term

and \( q \) is a factor of the leading coefficient.

**NOTE:** This theorem does not guarantee a zero. It only gives POSSIBLE rational zeros.

For each polynomial function, (a) list all possible rational zeros, (b) find all rational zeros, (c) factor \( f(x) \).

\[ #35 \quad f(x) = x^3 - 2x^2 - 13x - 10 \]

a. List all possible rational zeros:

- \( p \) is a factor of the constant term: constant term = -10
- possible \( p \): ±1, ±2, ±5, ±10
- \( q \) is a factor of the leading coefficient: leading coefficient = 1
- possible \( q \): ±1
- possible zeros = \( \frac{q}{p} \): ±1, ±2, ±5, ±10

b. Find all rational zeros.

\[
\begin{align*}
    f(1) &= -24 \\
    f(-1) &= 0 \\
    f(2) &= -36 \\
    f(-2) &= 0 \\
    f(5) &= 0 \\
\end{align*}
\]

Zeros = {-2, -1, 5}

No need to keep testing, since this function can have at most 3 zeros.

c. Factor \( f(x) \)

**NOTE:** For this function, after we found one zero, we could have used synthetic division and factoring to find the rest.

\[
\begin{array}{c|cccc}
-1 & 1 & -2 & -13 & -10 \\
 & -1 & 3 & 10 & \\
\hline
 & 0 & -3 & -10 & \\
\end{array}
\]

So

\[ f(x) = (x + 1)(x^2 - 3x - 10) \]

\[ f(x) = (x + 1)(x - 5)(x + 2) \]

Setting each factor equal to zero gives:

Zeros = {-2, -1, 5}

\[ #39 \quad f(x) = 6x^3 + 17x^2 - 31x - 12 \]

a. List all possible rational zeros:

- \( p \) is a factor of the constant term: constant term = -12
- possible \( p \): ±1, ±2, ±3, ±4, ±6, ±12
- \( q \) is a factor of the leading coefficient: leading coeff. = 6
- possible \( q \): ±1, ±2, ±3, ±6

\[ \frac{p}{q} = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \frac{1}{2}, \frac{3}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{6}, \frac{2}{6} \]
b. Find all rational zeros.

\[ f(1) = -20 \]
\[ f(-1) = 30 \]
\[ f(2) = 42 \]
\[ f(-2) = 70 \]
\[ f(3) = 210 \]
\[ f(-3) = 41 \]
\[ f(4) = 529 \]
\[ f(-4) = 0 \]

Etc...

-4 is a zero.

c. Factor \( f(x) \)

\[
\begin{array}{cccc}
6 & 17 & -31 & -12 \\
\hline
4 & -24 & 28 & 12 \\
6 & -7 & -3 & 0 \\
\end{array}
\]

So

\[ f(x) = (x - 4)(6x^2 - 7x - 3) \]
\[ = (x - 4)(3x + 1)(2x - 3) \]

Setting each factor equal to zero gives...

Zeros \[= \{ -4, -\frac{1}{3}, \frac{3}{2} \} \]

Use Descartes' Rule of signs to determine the possible number of positive real zeros and negative real zeros for each function.

Let \( f(x) \) be a polynomial function with real coefficients and a nonzero constant term, with terms in descending powers of \( x \).

- The number of positive real zeros of \( f(x) \) is either equal to the number of variations in sign occurring in the coefficients of \( f(x) \), or is less than the number of variations by a positive even integer.
- The number of negative real zeros of \( f(x) \) is either equal to the number of variations in sign occurring in the coefficients of \( f(-x) \), or is less than the number of variations by a positive even integer.

\[ f(x) = x^3 + 3x^4 - 3x - 2x + 3 \]

For \( f(x) \), number of sign variations = 2.

The number of positive real zeros = 2 or 0.

\[ f(-x) = (-x)^3 + 3(-x)^4 - (-x)^3 + (-x) + 3 \]
\[ = -x^3 + 3x^4 + x^3 - 2x + 3 \]

For \( f(x) \), number of sign variations = 3.

The number of negative real zeros = 3 or 1.
Polynomial Functions, 3.4

Here we consider graphs of the form $f(x) = ax^n$. For example, we have the graphs

Conclusions:
For positive and even exponent:
1. Shape: parabola-like
2. Domain = $(\mathbb{R}, \infty)$
   Range = $(0, \infty)$
3. For large exponents:
   - graph flattens near $(0, 0)$
   - is steeper at ends

Let's compare the graphs of
$f(x) = x^3, f(x) = x^5, f(x) = x^{21}$

Conclusions:
For positive and odd exponents:
1. Shape: cubic-like
2. Domain = $(\mathbb{R}, \infty)$
   Range = $(\mathbb{R}, \infty)$
3. For large exponents:
   - graph flattens near $(0, 0)$
   - is steeper at ends
Transformations to the graph still apply!

Recall: For an equation of the form \( f(x) = ax^2 \), \( a \) determines the width of the graph.
- \( |a| > 1 \) means the graph is vertically stretched
- \( |a| < 1 \) means the graph is vertically shrunk (or compressed)

Recall: For an equation of the form \( f(x) = a(x-h)^2 + k \), \( h \) and \( k \) determine translations of the graph.
- positive \( h \) value will shift the graph to the right
- negative \( h \) value will shift the graph to the left
- positive \( k \) value will shift the graph up
- negative \( k \) value will shift the graph down

Sketch the graph of each polynomial function

**#14** \( f(x) = -x^4 + 2 \)

Expect:
- parabola-like shape
- reflected over \( x \)-axis
- shifted up by 2

**#20** \( f(x) = \frac{1}{3}(x+1)^3 - 3 \)

Expect:
- cubic-like shape
- shifted left by 1
- shifted down by 3
- vertically shrunk (compressed)

Remember:
- Degree of polynomial = highest power on \( x \)
- A zero of a function = an \( x \)-value that solves \( f(x) = 0 \)
- an \( x \)-intercept: \((x, 0)\)
- A turning point = change from

Increasing to decreasing  Decreasing to increasing

The graph of a polynomial function with degree \( n \):
- Is continuous,

  Continuous means the hand-drawn graph can be sketched without lifting the pencil from the paper.
• Has smooth rounded turns, (No sharp corners or cusps.)
• Graph has at most \( n \) real zeros (\( x \)-intercepts)
  Number of Zeros Theorem mentioned last class
• Graph has at most \( n-1 \) turning points,
  at least one turning point between each successive pair of \( x \)-intercepts

\[ \begin{array}{c}
\text{Multiplicity of Zeros: the number of times a zero occurs}
\end{array} \]

Suppose that \( k \) is a zero of a polynomial function.
Consider the multiplicity of \( k \):

• If \( \text{multiplicity} = \text{one} \),
  then the graph crosses the \( x \)-axis at \((k, 0)\).

\[ \begin{array}{c}
\text{If multiplicity = one,}
\text{then the graph crosses the x-axis at (k, 0).}
\end{array} \]

• If \( \text{multiplicity} = \text{an even number} \),
  then the graph is tangent to the \( x \)-axis at \((k, 0)\).
(Tangent \( \rightarrow \) Graph touches the \( x \)-axis at one point only.)
(Gook says "graph bounces and turns at" \( k \).

\[ \begin{array}{c}
\text{If multiplicity = an even number,}
\text{then the graph is tangent to the x-axis at (k, 0).}
\end{array} \]

• If \( \text{multiplicity} = \text{an odd number} \) (greater than one),
  then the graph crosses AND is tangent to the \( x \)-axis at \((k, 0)\).
(Tangent to the \( x \)-axis and passes over.)
(Gook says "the graph wiggles at" \( k \).

\[ \begin{array}{c}
\text{If multiplicity = an odd number,}
\text{then the graph crosses AND is tangent to the x-axis at (k, 0).}
\end{array} \]

End Behavior:
• describes the ends of the graph
• determined by the dominating term

Dominating Term
= the term of the polynomial with highest degree, \( ax^n \)

Suppose that \( \alpha x^n \) is the dominating term of the polynomial function \( f \) of even degree.

Positive \( \alpha \)  
Negative \( \alpha \)
As \( x \to \infty \), \( f(x) \to -\infty \)
As \( x \to -\infty \), \( f(x) \to \infty \)

**NOTE:** If degree is even, then the shape is parabola-like.
Suppose that \( ax^n \) is the dominating term of a polynomial function \( f \) of odd degree.

Positive \( a \) \hspace{2cm} Negative \( a \)

As \( x \to \infty \), \( f(x) \to \infty \)
As \( x \to -\infty \), \( f(x) \to -\infty \)

**NOTE:** If degree is odd, then the shape is cubic-like.

Use a diagram to describe the end behavior of the graph.
Use an end behavior diagram to describe the end behavior of the graph of the polynomial function.

\#21) \( f(x) = 5x^5 + 2x^3 - 3x + 4 \)

Dominating Term = \( 5x^5 \)
Degree = 5 (odd)
Leading Coefficient \( a = 5 \) (positive)

\#27) \( f(x) = 3 + 2x - 4x^2 - 5x^{10} \)

Descending Order: \( f(x) = -5x^{10} - 4x^2 + 2x + 3 \)
Dominating Term = \(-5x^{10}\)
Degree = 10 (even)
Leading Coefficient \( a = -5 \) (negative)

**Graphing a Polynomial Function:**

Let \( f(x) = a_0x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \), be a polynomial function of degree \( n \).

**To sketch the graph:**

**Step 1:** Find the real zeros of \( f \). Plot \( x \)-intercepts.

**Step 2:** Find \( f(b) = 0 \). Plot \( y \)-intercept.

**Step 3:** Find the end behavior.

The end behavior starts at the outer most \( x \)-intercepts.

- Use the multiplicity of each zero to decide whether a graph crosses or touches the \( x \)-axis at the corresponding \( x \)-intercept.
- Find test points within intervals formed by the \( x \)-intercepts.

The sign (positive or negative) of the \( y \)-value will determine whether the graph is above or below the \( x \)-axis in that interval.

Graph the polynomial.
#32) \( f(x) = x^2(x + 1)(x-1) \)

Find the Zeros: \( x^2(x + 1)(x-1) = 0 \)

\[
\begin{align*}
x^2 &= 0 & x + 1 &= 0 & x - 1 &= 0 \\
x &= 0 & x &= -1 & x &= 1 \\
\end{align*}
\]

Zeros: \([-1, 0, 1]\)

x-intercepts: \((-1, 0), (0, 0), (1, 0)\)

Find the y-intercept: \( f(0) = 0^2(0 + 1)(0-1) \)

\[
f(0) = 0
\]

y-intercept: \((0, 0)\)

Find the End Behavior:

- Dominating term: \(x^4\)

<table>
<thead>
<tr>
<th>Zero</th>
<th>Mult.</th>
<th>Touch/Cross</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>1</td>
<td>Touch</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>Cross</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>Cross</td>
</tr>
</tbody>
</table>

Test Points:

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>-0.5</td>
<td>-0.1876</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.1876</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Graph the polynomial function.

Factor first if the expression is not in factored form.

#29) \( f(x) = x^3 + 5x^2 + 2x - 8 \)

Since we do not have a method to factor this directly, we must first find a zero and then use synthetic division to finish factoring.

Find the Zeros:

- **Rational Zeros Theorem:**
  - \(p\) is a factor of the constant term:
    - Constant term: \(-8\)
    - Possible \(p\): \(\pm 1, \pm 2, \pm 4, \pm 8\)
  - \(q\) is a factor of the leading coefficient:
    - Leading coeff.: \(1\)
    - Possible \(q\): \(\pm 1\)
  - Possible zeros: \(\frac{p}{q} = \pm 1, \pm 2, \pm 4, \pm 8\)

- **Descartes' Rule of Signs:**
  - \(f(x) = x^3 + 5x^2 + 2x - 8\)
  - For \(f(x)\), number of sign variations = 1.
  - The number of positive real zeros = 1.
\[
f(-x) = (-x)^3 + 5(-x)^2 + 2(-x) - 8
\]
\[
= -x^3 + 5x^2 - 2x - 8
\]

For \( f(-x) \), number of sign variations = 2.

The number of negative real zeros = 2 or 0.

Using these rules together, it would be wise to try to find a positive zero. The possibilities include +1, +2, +4, +8.

Use synthetic division:

\[
\begin{array}{c|cccc}
1 & 1 & 5 & 2 & -8 \\
\hline
 & 1 & 6 & 8 & 0 \\
\end{array}
\]

This implies 1 is a zero: (i.e. \( x - 1 \) is a factor)

Since we know (via Descartes’ Rule) that there is only one positive zero, we can stop testing once we've found this one.

Notice the other 2 zeros (there are a possible maximum number of 3 since degree is 3) must be negative or complex.

Now that we've found one zero, we can use the quotient (just calculated with synthetic division) to finish factoring.

Find the Zeros:

\[
f(x) = (x-1)[x^2 + 6x + 8]
\]
\[
= (x-1)(x+2)(x+4)
\]
\[
(x-1)(x+2)(x+4) = 0
\]
\[
x = 1 \quad x = -2 \quad x = -4
\]

Zeros = \(-4, -2, 1\)

\(x\)-intercepts: \((-4, 0), (-2, 0), (1, 0)\)

Find the \(y\)-intercept:

\[
f(0) = (0-1)(0+2)(0+4)
\]
\[
f(0) = -8
\]

\(y\)-intercept = \((0, -8)\)

Find the End Behavior:

\[
\text{dominating term} = x^3
\]

<table>
<thead>
<tr>
<th>Zero</th>
<th>Mult.</th>
<th>Touch/Cross</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>1</td>
<td>CROSS</td>
</tr>
<tr>
<td>-2</td>
<td>1</td>
<td>CROSS</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>CROSS</td>
</tr>
</tbody>
</table>

Test Points:

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>0</td>
</tr>
<tr>
<td>-3</td>
<td>4</td>
</tr>
<tr>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>-8</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
The Factor Theorem: (restated)

If \( a \) is an \( x \)-intercept of the graph of \( y = f(x) \), then
- \( a \) is a zero of \( f \)
- \( a \) is a solution of \( f(x) = 0 \), and
- \( x - a \) is a factor of \( f(x) \).

Find a polynomial of least possible degree having the graph shown.

\[ #65 \]

\[ f(x) = a(x + 6)(x - 2)(x - 5) \]
\[ f(0) = a(0 + 6)(0 - 2)(0 - 5) \]
\[ 30 = a(60) \]
\[ 0.5 = a \]
\[ f(x) = 0.5x^3 - 0.5x^2 - 16x + 30 \]

Find a polynomial of least possible degree having the graph shown.

\[ #70 \]

Consider the way the graph interacts with the \( x \)-axis. What is the multiplicity of each zero?
(Keep in mind that we want a polynomial of least degree possible.)

\[ f(x) = a(x + 1)^2(x - 2) \]
\[ f(0) = a(0 + 1)^2(0 - 2) \]
\[ 4 = a(-2) \]
\[ -2 = a \]
\[ f(x) = -2x^2 + 6x + 4 \]

Intermediate Value Theorem:

Let \( f(x) \) be a polynomial function with only real coefficients.
Let \( a \) and \( b \) be real numbers.
If the values \( f(a) \) and \( f(b) \) are opposite in sign, then there exists at least one real zero between \( a \) and \( b \).

Use the intermediate value theorem for polynomials to show that each polynomial function has a zero between the numbers given.
#55) \( f(x) = x^3 - 4x^2 - 20x + 23 \); \(-1\) and \(0\)

\[
f(-1) = (-1)^3 - 4(-1)^2 - 20(-1) + 23 = -1 - 4 - 20 + 23 = 24
\]

\(f(-1)\) is negative

\[
f(0) = (0)^3 - 4(0)^2 - 20(0) + 23 = 23
\]

\(f(0)\) is positive

The IVT states that \(f(x)\) will have a zero between \(-1\) and 0.

**Boundedness Theorem:**

Let \(f(x)\) be a polynomial function of degree \(n \geq 1\)

- with real coefficients and
- with a positive leading coefficient.

Divide \(f(x)\) by \(x - c\) using synthetic division.

If \(c > 0\) and all numbers in the bottom row of the synthetic division are nonnegative, then \(f(x)\) has no zero greater than \(c\).

If \(c < 0\) and the numbers in the bottom row of the synthetic division alternate in sign (with 0 considered positive or negative, as needed), then \(f(x)\) has no zero less than \(c\).

Show that the real zeros of each polynomial function satisfy the given conditions.

#61) \( f(x) = 3x^4 + 2x^3 - 4x^2 + 1 - 1 \); no real zero greater than 1

\[
\begin{array}{cccc}
1 & 3 & 2 & -4 & 1 & -1 \\
\hline
 & 3 & 5 & 1 & 2
\end{array}
\]

All numbers in bottom row are non-negative.

Boundedness Theorem states that \(f(x)\) will have no real zeros greater than 1.

Notice: 1 is NOT a zero of \(f(x)\). Why?

- Remainder is not zero.

#64) \( f(x) = x^5 - 3x^3 + x + 2 \) no real zero less than \(-3\)

\[
\begin{array}{cccccc}
-3 & 1 & 0 & -3 & 0 & 1 & 2 \\
\hline
 & -3 & 9 & -18 & 54 & -165
\end{array}
\]

Numbers in bottom row alternate in sign.

Boundedness Theorem states that \(f(x)\) will have no real zeros less than \(-3\).
Week 9

Textbook Section 3.4 Continued
Zeros of Polynomial Functions

Objectives

- The student will be able to use the theorems learned in sections 3.3 and 3.4 to identify key characteristics of a higher order polynomial
- The student will be able to use the theorems learned in sections 3.3 and 3.4 to graph a higher order polynomial

Key Concepts from 3.3

Polynomial function:

\[ f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \]

A factor \( x - k \) of a polynomial divides evenly.

Setting these factors equal to zero yields the zeros.

Fundamental Theorem of Algebra:

Every function defined by a polynomial function of degree one or more has at least one complex zero.

Number of Zeros Theorem:

A polynomial function of degree \( n \) has at most \( n \) distinct zeros.

The number of times a zero occurs is called the multiplicity.

Factor Theorem:

The polynomial \( x - k \) is a factor of the polynomial \( f(x) \) if and only if \( f(k) = 0 \).

Conjugate Zeros Theorem:

Let \( f(x) \) be a polynomial function having only real coefficients. If \( z = a + bi \) is a zero of \( f(x) \), then \( z = a - bi \) is a zero of \( f(x) \).

Rational Zeros Theorem:

Let \( f(x) \) be a polynomial function with integer coefficients.

Let \( \frac{p}{q} \) be a rational number written in lowest terms.

If \( \frac{p}{q} \) is a zero of \( f \), then \( p \) is a factor of the constant term and \( q \) is a factor of the leading coefficient.

NOTE: This theorem does not guarantee a zero. It only provides POSSIBLE rational zeros.

Descartes’ Rule of Signs:

Let \( f(x) \) be a polynomial function with real coefficients and a nonzero constant term, with terms in descending powers of \( x \).

- The number of positive real zeros of \( f \) either equals the number of variations in sign occurring in the coefficients of \( f(x) \), or is less than the number of variations by a positive even integer.
- The number of negative real zeros of \( f \) either equals the number of variations in sign occurring in the coefficients of \( f(-x) \), or is less than the number of variations by a positive even integer.

NOTE: Real zeros are not necessarily rational.

Key Concepts from 3.4

Graphs of the form: for \( n > 0 \)

Even exponent \[ \rightarrow \] Shape: parabola-like
Odd exponent \[ \rightarrow \] Shape: cubic-like
For large exponents, \[ \rightarrow \] graph flattens near \((0, 0)\)
is steeper at ends

A turning point is a change in graph from increasing to decreasing or decreasing to increasing.
The graph of a polynomial function with degree \( n \).

- Is continuous,
- Has smooth rounded turns,
- Has at most \( n \) real zeros (x-intercepts),
- Has at most \( n-1 \) turning points,
- Has at least one turning point between each successive pair of x-intercepts.

**Multiplicity of Zeros:**

The number of times a zero occurs

Suppose that \( k \) is a zero of a polynomial function.
Consider the multiplicity of \( k \):

- If multiplicity = one, then graph crosses x-axis at \((k, 0)\)
- If multiplicity = an even number, then graph is tangent to x-axis at \((k, 0)\).
- If multiplicity = an odd number (greater than one), then the graph crosses AND is tangent to x-axis at \((k, 0)\).

**End behavior:**

Suppose \( ax^n \) is the dominating term of a polynomial function \( f \).

\[
\begin{align*}
\text{positive } a, \text{ even degree} & \quad \text{positive } a, \text{ odd degree} \\
\text{negative } a, \text{ even degree} & \quad \text{negative } a, \text{ odd degree}
\end{align*}
\]

**NOTE:** If degree is even, then the shape is parabola-like.
**NOTE:** If degree is odd, then the shape is cubic-like.

**To graph a polynomial function:**

**Step 1:** Find the real zeros of \( f \).
**Step 2:** Find \( f(0) \).
**Step 3:** Find the end behavior.
  - Use the multiplicity of each zero.
  - Find test points.

**The Factor Theorem:** (restituted)

If \( a \) is an x-intercept of the graph of \( y = f(x) \), then

- \( a \) is a zero of \( f \),
- \( a \) is a solution of \( f(a) = 0 \), and
- \( x - a \) is a factor of \( f(x) \).

**Intermediate Value Theorem:**

Let \( f(x) \) be a polynomial function with only real coefficients.
Let \( a \) and \( b \) be real numbers.

If the values \( f(a) \) and \( f(b) \) are opposite in sign, then there exists at least one real zero between \( a \) and \( b \).

**Boundedness Theorem:**

Let \( f(x) \) be a polynomial function of degree \( n \geq 1 \)

- with real coefficients and
- with a positive leading coefficient.

Divide \( f(x) \) by \( x - c \) using synthetic division.

- If \( c > 0 \) and all numbers in the bottom row of the synthetic division are nonnegative, then \( f(x) \) has no zero greater than \( c \).
- If \( c < 0 \) and the numbers in the bottom row of the synthetic division alternate in sign (with 0 considered positive or negative, as needed), then \( f(x) \) has no zero less than \( c \).
Online Homework and Quiz Assignments

After completing the Key Concepts and Handouts, log into MyLabsPlus and begin your homework and quiz for this week, go to www.ucf.mylabsplus.com and begin working on your assignments.
Week 10

Textbook Section 3.5
Rational Functions: Graphs, Applications, Models

Objectives

- The student will be able to analyze graphs of rational functions
- The student will be able to graph rational functions using transformations
- The student will be able to find vertical, horizontal, and oblique asymptotes
- The student will be able to sketch a graph of the rational function
- The student will be able to find an equation for a rational function given certain features of the graph

Key Concepts

Definition:

A rational function $f$ of the form

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials, with $q(x) \neq 0$.

Transformations of a graph apply to rational functions too.

Vertical Asymptote:

A vertical line $x = a$ which the graph approaches but does not cross. Occurs when $|f(x)| \to \infty$ as $x \to a$.

To find vertical asymptotes:

- Factor and reduce the fraction.
- Set the denominator equal to 0 and solve for $x$.
- If $a$ is a zero of the denominator, then the line $x = a$ is a vertical asymptote.

Horizontal Asymptote:

A horizontal line ($y = b$) which the graph approaches.

A graph may cross its horizontal asymptote.

Occurs when $f(x) \to b$ as $|x| \to \infty$.

To find horizontal asymptotes: (3-part rule)

Find the degree of numerator and denominator. Then compare.

<table>
<thead>
<tr>
<th>Degree relationship</th>
<th>Horizontal asymptote</th>
</tr>
</thead>
<tbody>
<tr>
<td>degree num &lt; degree denom</td>
<td>$y = 0$</td>
</tr>
<tr>
<td>degree num = degree denom</td>
<td>$y = \text{leading coefficient}$</td>
</tr>
<tr>
<td>degree num &gt; degree denom</td>
<td>none</td>
</tr>
</tbody>
</table>

Oblique Asymptotes:

- a slanted line that the graph approaches
- will occur when the degree of the numerator is exactly 1 more than the degree of the denominator

To find an oblique asymptote:

- divide the numerator by the denominator and disregard the remainder (using synthetic or long division)
- set the quotient equal to $y$ to obtain the equation of the asymptote

To sketch the graph:
Step 1: Find any vertical asymptotes.
Step 2: Find any horizontal or oblique asymptotes.
Step 3: Find the y-intercept by evaluating f(0).
Step 4: Find the x-intercepts (the zeros), by solving f(x) = 0.
Step 5: Determine whether the graph will intersect its non-vertical asymptote.
   \[ \text{For horizontal asymptote, } y = b \quad \text{solve } f(x) = b \]
   \[ \text{For oblique asymptote, } y = mx + b \quad \text{solve } f(x) = mx + b \]
Step 6: Plot test points.
Step 7: Complete the sketch.

A “Hole” in the Graph:

If a rational function can be reduced as a fraction, the eliminated factor yields
   • an x-value that is not part of the domain and
   • causes an open “hole” in the graph.

Textbook Section 3.6
Variation

Objectives

- The student will be able to solve direct, inverse and combined variation problems

Key Concepts

Direct Variation:

\[ y \text{ varies directly as } x \text{, or } y \text{ is directly proportional to } x, \text{ if there exists a nonzero real number } k \text{, called the constant of variation, such that } y = kx \]

Direct Variation as nth power:

\[ y = kx^n \]

Inverse Variation as nth Power:

\[ y = \frac{k}{x^n} \]

Combined Variation:

\[ y = kx^n z^m \text{ (joint variation)} \]

\[ OR \quad y = \frac{kx^n}{z^m} \]

Solving Variation Problems:

Step 1: Define your variables. Write an equation using one of the variations listed and the constant k
Step 2: Substitute the given values of the variables. Find the value of k
Step 3: Substitute this value of k into the original equation to obtain a specific formula.
Step 4: Solve the problem using the equation you’ve created.

Online Homework and Quiz Assignments

After completing the Key Concepts and Handouts, log into MyLabsPlus and begin your homework and quiz for this week, go to www.uct.mylabsplus.com and begin working on your assignments.
Graphing a Rational Function, 3.5

Recall: A rational function is defined by

\[ f(x) = \frac{p(x)}{q(x)} \]

where \( p(x) \) and \( q(x) \) are polynomials and the rational expression is written in lowest terms.

To sketch the graph:

- **Step 1**: Find any vertical asymptotes.
- **Step 2**: Find any horizontal or oblique asymptotes.
- **Step 3**: Find the y-intercept by evaluating \( f(0) \).
- **Step 4**: Find the x-intercepts, if any, by solving \( f(x) = 0 \) (These will be the zeros of the numerator, \( p(x) \)).
- **Step 5**: Determine whether the graph will intersect its non-vertical asymptote.
- **Step 6**: Find test points.
- **Step 7**: Complete the sketch.

**Example: #61**

Sketch the graph of the rational function.

\[ f(x) = \frac{x+1}{x-4} \]

Solution:

- **Step #1**: Vertical asymptotes occur whenever the denominator is equal to 0:
  \[ x - 4 = 0 \]
  \[ x = 4 \]

- **Step #2**: Since the degree of the numerator equals the degree of the denominator, the horizontal asymptote is given by
  \[ y = \frac{\text{leading term of } p(x)}{\text{leading term of } q(x)} = \frac{1}{1} = 1 \]

  There are no oblique asymptotes in this case.

- **Step #3**: To find the y-intercept, we evaluate \( f(0) \):
  \[ f(0) = \frac{0+1}{0-4} = -\frac{1}{4} \]
  Therefore, the point \( (0, -\frac{1}{4}) \) is on the graph.

- **Step #4**: To find the x-intercepts, if any, we solve \( f(x) = 0 \):
  \[ 0 = \frac{x+1}{x-4} \]
  \[ 0 = x + 1 \]
  \[ x = -1 \]
  Therefore, the point \( (-1, 0) \) is on the graph.

- **Step #5**: Since we have a horizontal asymptote, we want to know if the graph will ever cross it. We try to solve
  \[ HA = 1 = \frac{x+1}{x-4} \]
  but this implies that \( 1 = -4 \), which is always false. Hence, the graph never crosses the horizontal asymptote.

- **Step #6**: In finding test points, we pick test points on either side of the vertical asymptote \( x = 4 \). Here, we try \( x = -1, 5 \).
  \[ f(-1) = 0 \]
  \[ f(5) = 6 \]
Step #7.

Now, we plot everything we know so far:

To complete the graph, we remember that rational functions will "hug" their asymptotes:

Example: #67)

Sketch the graph of the rational function.

\[ f(x) = \frac{3x}{x^2 - x - 2} \]

Solution:

To find the vertical asymptotes, we set the denominator equal to 0, which gives us the two solutions \( x = 2, -1 \).

We note that the degree of the numerator, which is 1, is less than the degree of the denominator, which is 2, so the horizontal asymptote is \( y = 0 \). Also, there is no oblique asymptote.

To find the \( x \)-intercepts, we set the numerator equal to 0:

\[ 3x = 0 \text{ implies that } x = 0 \]

so the point \((0,0)\) is on the graph. To find the \( y \)-intercept, we calculate \( f(0) \), but we have just found that this is equal to 0! Hence, \((0,0)\) is the \( x \)- and \( y \)-intercept. We also note that the graph will cross the horizontal asymptote at \((0, 0)\).

We pick the test points \( x = -2, -0.5, 1, 3 \).

We plot everything we know so far.
and complete the graph, remembering to hug the horizontal asymptote $y = 0$, which is not graphed because it coincides with the axis.

Example: #73)

Sketch the graph of the rational function.

$$f(x) = \frac{3x^2 + 3x - 6}{x^2 - x - 12}$$

Solution:

First, we factor the numerator and denominator:

$$f(x) = \frac{3(x+2)(x-1)}{(x-4)(x+3)}$$

Since nothing cancels, we continue on as usual. To find the vertical asymptotes, we set the denominator equal to 0, which yields the two asymptotes $x = 4, -3$.

Since the degree of the numerator is equal to the degree of the denominator, the horizontal asymptote is given by

$$y = \frac{\text{leading term of numerator}}{\text{leading term of denominator}} = \frac{3}{1} = 3$$

There is no oblique asymptote.

To find the $x$-intercept(s), we set the numerator equal to 0, which yields $x = -2, 1$, so that $(-2, 0)$ and $(1, 0)$ are on the graph. To find the $y$-intercept, we set $x = 0$, which yields $y = \frac{3}{1}$, so the point $(0, 3)$ is on the graph.

Since we have a horizontal asymptote, we want to know if we ever cross it. We solve

$$\text{HA} = 3 = \frac{3x^2 + 3x - 6}{x^2 - x - 12}$$

which has the solution $x = -5$, so $(-5, 3)$ is on the graph. Finally, we use the test points $x = -4, -2, 0, 1, 7$.

We plot everything we know.
Example: #88)

Sketch the graph of 
\[ f(x) = \frac{2x^2 + 3}{x^2 - 4} \]

Solution:

The vertical asymptote is found by setting the denominator equal to 0, yielding \( x = 4 \). Since the degree of the numerator is one more than the degree of the denominator, there is no horizontal asymptote, but there is an oblique asymptote. To find this, we use synthetic division and discard the remainder:

\[
\begin{array}{c|ccc}
4 & 2 & 0 & 3 \\
\hline
& & 8 & 32 \\
\hline
& & 2 & 8 & 35 \\
\end{array}
\]

Hence, the oblique asymptote is given by \( y = 2x + 8 \).

To find the \( x \)-intercepts, we set the numerator equal to 0, but this only has imaginary solutions, so there are no \( x \)-intercepts. To find the \( y \)-intercept, we set \( x = 0 \), which yields \( y = -0.75 \). For our test points, we choose \( x = 2, 5 \). We plot what we know:
And we finish the graph:

where we make sure to hug the oblique asymptote.

A 'Hole' in the Graph: If a rational function can be reduced as a fraction, the eliminated factor yields
• an $x$-value that is not part of the domain and
• causes an open "hole" in the graph.

Example: #95)

Sketch the graph of the rational function.

$$r(x) = \frac{x^2 - 1}{x^2 - 4x + 3}$$

Solution:

We begin by factoring and canceling:

$$f(x) = \frac{x^2 - 1}{x^2 - 4x + 3} = \frac{(x-1)(x+1)}{(x-3)(x-1)} = \frac{x+1}{x-3}, \quad x \neq 1$$

where we note that we have a restriction in our domain. It may appear that we can calculate $f(1)$ after reducing, which would equal $-1$, but we have to remember that there is a hole in the graph at $(1, -1)$.

The vertical asymptote is given by $x = 3$ and the horizontal asymptote is given by $y = 1$. There are no oblique asymptotes. We can show that the graph does not intersect the horizontal asymptote.

To find the $y$-intercept, we set $x = 0$, yielding $y = -1/3$. The $x$-intercept is given by setting $y = 0$, which yields the two solutions $x = -1$ and $1$. However, we recall that there is a hole in the graph at $x = 1$, so we only have the $y$-intercept $x = -1$.

Finally, we choose the test points $x = 2, 1, 4$. We graph everything we know:
Example: #92)

Sketch the graph of the rational function.

\[ f(x) = \frac{x^2 - 16}{x - 4} \]

Solution:

We first factor and cancel:

\[ f(x) = \frac{(x+4)(x-4)}{x-4} = x + 4, \quad x \neq 4 \]

This is just a line with a hole at \( x = -4 \).
Variation, 3.6

Direct Variation: We say that "y varies directly as x," or "y is directly proportional to x," if there exists a nonzero real number k, called the constant of variation, such that

\[ y = kx \]

Solving Variation Problems:

- Step 1 Define your variables. Write an equation using the variation listed and the constant k.
- Step 2 Substitute the given values of the variables. Find the value of k.
- Step 3 Substitute this value of k into the original equation to obtain a specific formula.
- Step 4 Solve the problem using the equation you've created.

Example: #11)

If y varies directly as x, and y = 20 when x = 4, find y when x = -6.

Solution:

We read "y varies directly as x" to mean

\[ y = kx \]

and know that y = 20 when x = 4 so we can solve for k:

\[ 20 = 4k \]

so that k = 5. Now, we would like to find y when x = -6:

\[ y = 5(-6) = -30 \]

Occasionally, you will see "y varies directly as the nth power of x" or "y is directly proportional to the nth power of x," which means that

\[ y = kx^n \]

You will also see "y varies inversely as the nth power of x" or "y is inversely proportional to the nth power of x," which means that

\[ y = \frac{k}{x^n} \]

There is also joint variation. If "y varies jointly with x and z," then

\[ y = kxz \]

Example: #13)

If m varies jointly as x and y, and m = 10 when x = 2 and y = 14, find m when x = 11 and y = 8.

Solution:

Since m varies jointly with x and y, we write

\[ m = kxy \]

and plug in what we know:

\[ 10 = 2(14) \]

\[ k = \frac{5}{14} \]

and we have the equation

\[ m = \frac{5}{14}xy \]

Given the new values for x and y, we solve for the new m:

\[ m = \frac{5}{14}(11)(8) = \frac{220}{7} \approx 31.429 \]

Example: #15)

If y varies inversely as x, and y = 10 when x = 3, find y when x = 20.

Solution:

Since y varies inversely with x, we write

\[ y = \frac{k}{x} \]
and plug in to find

\[ 10 = \frac{k}{3} \]

so that \( k = 30 \). To find the new value of \( y \), we then plug in:

\[ y = \frac{30}{20} = \frac{3}{2} = 1.5 \]

Example: #18)

Suppose \( p \) varies directly as the square of \( z \), and inversely as \( r \). If \( p = \frac{32}{5} \) when \( z = 4 \), and \( r = 10 \), find \( p \) when \( z = 3 \) and \( r = 36 \).

Solution:

We have the variation equation

\[ p = \frac{k z^2}{r} \]

and plug in what we know to find \( k \):

\[ \frac{32}{5} = k \frac{4^2}{10} \]

so that \( k = 4 \). Then we plug in the new values:

\[ p = \frac{4 \cdot 3^2}{36} = \frac{36}{36} = 1 \]

Example: #26)

The amount of water emptied by a pipe varies directly as the square of the diameter of the pipe. For a certain constant water flow, a pipe emptying into a canal will allow 200 gal of water to escape in an hour. The diameter of the pipe is 6 in.

How much water would a 12 in. pipe empty into the canal in an hour, assuming the same water flow?

Solution:

First we define our variables:

\( A = \) amount of water emptied

\( D = \) diameter of pipe

Then we read

\[ A = kD^2 \]

and solve for \( k \):

\[ 200 = k \cdot 6^2 \]

so that \( k = \frac{200}{36} \). Then we plug in the new value for \( D \):

\[ A = \frac{50}{9} \cdot (12)^2 = \frac{50}{9} \cdot 144 = 800 \]

A 12 in. pipe would allow 800 gallons of water to empty into the canal in an hour.
Week 11

Textbook Section 4.1
Inverse Functions

Objectives

- The student will be able to decide whether a function is one-to-one
- The student will be able to show that two functions are inverses of one another
- The student will be able to find the inverse of a one-to-one function
- The student will be able to graph the inverse of a one-to-one function
- The student will be able to evaluate an inverse function at an \( x \)-value given a list of function values

Key Concepts

One-to-one functions:

For a one-to-one function,

- each \( x \)-value corresponds to ONLY one \( y \)-value, and
- each \( y \)-value corresponds to ONLY one \( x \)-value.

Definition:

A function \( f \) is a one-to-one function if, for elements \( a \) and \( b \) in the domain of \( f \), \( a \neq b \) implies \( f(a) \neq f(b) \).

Tests that a function is one-to-one:

1. Show that \( f(a) = f(b) \) implies \( a = b \). (Two different \( y \)-values must originate from two different \( x \)-values.)
2. Every \( y \)-value corresponds to exactly one \( x \)-value.
   To show that a function is not one-to-one, find at least two \( x \)-values that produce the same \( y \)-value.
3. Sketch the graph and use the horizontal line test.
   If any horizontal line intersects the graph of a function in no more than one point, then the function is one-to-one.
4. If the function either increases or decreases on its entire domain, then it is one-to-one.

A function MUST be one-to-one in order to have an inverse.

Definition:

Let \( f \) be a one-to-one function.

The function \( g \) is called the inverse function of \( f \) if \( (f \circ g)(x) = x \) for every \( x \) in the domain of \( g \), and \( (g \circ f)(x) = x \) for every \( x \) in the domain of \( f \).

NOTATION: We denote the inverse function of \( f \) as \( f^{-1}(x) \)

Finding the inverse:

- Check that \( f(x) \) is one-to-one.
- Interchange \( x \) and \( y \).
- Solve for \( y \): Replace \( y \) with \( f^{-1}(x) \).
- To check, show \( (f \circ f^{-1})(x) = x \) and \( (f^{-1} \circ f)(x) = x \).

Facts About Inverses:

1. If \( (a, b) \) is a point on the graph of \( f(x) \), then \( (b, a) \) is a point on the graph of \( f^{-1}(x) \).
2. Domain of \( f(x) \) = Range of \( f^{-1}(x) \)
   Range of \( f(x) \) = Domain of \( f^{-1}(x) \)
3. The graph of \( f^{-1}(x) \) is the reflection of \( f(x) \) over the line \( y = x \).
Textbook Section 4.2
Exponential Functions

Objectives

- The student will be able to evaluate exponential functions
- The student will be able to graph basic exponential functions
- The student will be able to graph exponential functions using transformations
- The student will be able to write an equation for a given graph
- The student will be able to solve exponential equations
- The student will be able to solve application problems

Key Concepts

Exponential function

A function where the input, $x$, is an exponent

Properties of Exponents:

- Product Rule: $a^m \cdot a^n = a^{m+n}$
- Power Rule 1: $(a^m)^n = a^{mn}$
- Power Rule 2: $(ab)^n = a^n \cdot b^n$
- Power Rule 3: \( \left( \frac{a}{b} \right)^n = \frac{a^n}{b^n} \), \( b \neq 0 \)
- Zero Exponent Rule: $a^0 = 1$ \( (a \neq 0) \)

More Properties of Exponents:

Suppose $a$ is any real number, $a > 0$, $a \neq 1$. Then...

- $a^x$ is a unique real number for all real numbers $x$
- $a^x = a^y$ if and only if $b = c$
- If $a > 1$ and $m < n$, then $a^m < a^n$
- If $0 < a < 1$ and $m < n$, then $a^m > a^n$

Euler's Number: $e$

$e \approx 2.7182818284$

Definition:

An exponential function with base $a$ is defined by $f(x) = a^x$ where $a > 0$ and $a \neq 1$

Characteristics of the exponential graph:

- $f(a) = a^x$
- Always contains the points: $(-1, \frac{1}{a})$, $(0, 1)$, and $(1, a)$
- If $a > 1$, then $f$ is an increasing function.
  - If $0 < a < 1$, then $f$ is a decreasing function.
- The $x$-axis is a horizontal asymptote.
- Domain = $(-\infty, \infty)$, Range = $(0, \infty)$

Compound Interest:

$A = P \left(1 + \frac{r}{n}\right)^{nt}$

Interest compounded (or paid) a number of times per year

- $A =$ future value
- $P = $ present value
- $r =$ annual interest rate (as a decimal)
- $n =$ number of times per year interest is being added
- $t =$ time (years)

Continuous Compounding:
\[ A = Pe^{rt} \]

Interest compounded continuously

- \( A \) = future value
- \( P \) = present value
- \( r \) = annual interest rate (as a decimal)
- \( t \) = time (years)

---

**Online Homework and Quiz Assignments**

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Inverse Functions, 4.1

Given a function, we would like to know if it has an inverse. However, not all functions have inverses. We have the following definition:

For a one-to-one function, each x-value corresponds to only one y-value and each y-value corresponds to only one x-value.

Here we have an example of a function:

![Diagram showing a non-one-to-one function]

For instance, we know that \( f(2) = 7 \), \( f(4) = 8 \), etc. This function is not one-to-one, because \( f(2) = f(3) \).

This function, however, is one-to-one. Most of the time, however, we won't have these sorts of images. We have another useful definition:

A function \( f \) is a one-to-one function if, for elements \( a \) and \( b \) in the domain of \( f \), \( a \neq b \) implies \( f(a) \neq f(b) \).

In other words, if two y-values are the same, the x-values must also be the same. This gives us the equation test.

Assume that \( f(a) = f(b) \) and simplify. If this implies that \( a = b \), then the function is one-to-one. If not, then the function isn't one-to-one.

Example: #11

Decide whether \( f(x) = \sqrt{36 - x^2} \) is one-to-one.

Solution:

We calculate

\[
\begin{align*}
    f(a) &= \sqrt{36 - a^2} \\
    f(b) &= \sqrt{36 - b^2}
\end{align*}
\]

set them equal to each other, then square both sides to simplify:

\[
\begin{align*}
    \sqrt{36 - a^2} &= \sqrt{36 - b^2} \\
    36 - a^2 &= 36 - b^2 \\
    a^2 &= b^2
\end{align*}
\]

Now, can we necessarily conclude that \( a = b \)? Not necessarily. For instance, if \( a = -1 \), \( b = 1 \) then \( a^2 = b^2 \) but \( a \neq b \). We can take the square root of both sides to conclude that

\[
    a = \pm b
\]

but this is not enough. We need \( a = b \). Therefore, this function is not one-to-one.

Example: #11

Decide whether \( f(x) = 3x^2 - 6 \) is one-to-one.

Solution:

We set

\[
\begin{align*}
    f(a) &= f(b) \\
    3a^2 - 6 &= 3b^2 - 6 \\
    3a^2 &= 3b^2 \\
    a^2 &= b^2 \\
    a &= b
\end{align*}
\]
where in the last step we've taken the cube root of both sides. No $z$ is necessary since we're dealing with an odd root. Therefore, since we have concluded that $a = b$, this is a one-to-one function.

Occasionally, we will only have a graph available. Therefore, we define the horizontal line test:

If any horizontal line intersects the graph of a function in no more than one point, then the function is one-to-one.

**Example: #11**

Decide whether $y = 3x^2 - 6$ is one-to-one.

**Solution:**

When we graph this function, we note that any horizontal line only intersects the graph once. Hence, the function is one-to-one.

![Graph of $y = 3x^2 - 6$]

**Example: #17**

Decide whether $y = 2(x + 1)^2 - 6$ is one-to-one.

**Solution:**

When we graph the function, we note that a number of horizontal lines intersect the graph twice, so the function is not one-to-one.

![Graph of $y = 2(x + 1)^2 - 6$]

**Definition.** Let $f$ be a one-to-one function. The function $g$ is called the inverse function of $f$ if

- $(f \circ g)(x)$ for every $x$ in the domain of $g$
- $(g \circ f)(x)$ for every $x$ in the domain of $f$

in other words, $f$ "undoes" $g$ and $g$ "undoes" $f$. If $g$ is the inverse of $f$, we sometimes write

$$g(x) = f^{-1}(x)$$

**Example: #41**

Use the definition to determine whether $f$ and $g$ are inverses:

- $f(x) = 2x + 4$
- $g(x) = \frac{1}{2}x - 2$

**Solution:**

We calculate

$$\left( f \circ g \right)(x) = f(g(x)) = f\left( \frac{1}{2}x - 2 \right) = x - 4 + 4 = x$$

$g \circ f (x)$ can be calculated similarly.
\[
(g \circ f)(x) = g(f(x)) = g(2x + 4) = \\
= \frac{1}{3}(2x + 4) - 2 \\
= x + 2 - 2 \\
= x
\]

Therefore, the two functions are inverses.

**Example: #49**

Use the definition to determine whether \( f \) and \( g \) are inverses:

\( f(x) = x^2 + 3 \), domain \([0, \infty)\)

\( g(x) = \sqrt{x-3} \), domain \([3, \infty)\)

Solution:

If we don't have the restricted domain on \( f \), it is simply a parabola and does not pass the horizontal line test. In other words, \( f \) is not one-to-one, so it has no inverse.

However, by restricting the domain, we obtain the next graph. Essentially, we have cut off half of the parabola. By restricting the domain, the function \( f \) does pass the horizontal line test and should be expected to have an inverse.

We need to check that \( g \) is in fact this inverse.

We calculate

\[
(f \circ g)(x) = f(g(x)) = f(\sqrt{x-3}) = \\
= (\sqrt{x-3})^2 + 3 \\
= x - 3 + 3 \\
= x
\]

\[
(g \circ f)(x) = g(f(x)) = g(x^2 + 3) = \\
= \sqrt{x^2 + 3 - 3} \\
= \sqrt{x^2} = x
\]

In this last step, note that we have only taken the positive square root. When you're dealing with a function, the function will specify which root to take. If you're solving an equation, however, and need to take the square root of both sides, you must take both the positive and negative roots.

Therefore, these functions are inverses.

To find the inverse:

- Check that \( f(x) \) is one-to-one.
- Replace \( f(x) \) with \( y \).
- Interchange \( x \) and \( y \).
- Solve for \( y \).
- Replace \( y \) with \( f^{-1}(x) \).

If you want to check, show \( (f \circ f^{-1})(x) = x \) and \( (f^{-1} \circ f)(x) = x \).
Example: #55)
Write an equation for the inverse function in the form \( y = f^{-1}(x) \) where
\[
f(x) = 3x - 4
\]

Graph \( f \) and its inverse on the same axes and give their domains and ranges.

Solution:
We begin with the original function, replace \( f(x) \) with \( y \), interchange the variables \( x \) and \( y \), then solve for \( y \):
\[
\begin{align*}
y &= 3x - 4 \\
x &= 3y - 4 \\
x + 4 &= 3y \\
\frac{x + 4}{3} &= y
\end{align*}
\]
Then we replace \( y \) with \( f^{-1}(x) \)
\[
f^{-1}(x) = \frac{x + 4}{3}
\]

We then graph \( f \) (blue) and its inverse (red) on the same axes:

We have graphed the line \( y = x \) (dashed). Remember that inverses, when graphed, should be reflections of each other across this line, which is the case here.

To find the domains and ranges of these functions, we note that they are both linear functions, so their domains and ranges are all \((-\infty, \infty)\).

Facts About Inverses:
1. If \((a, b)\) is a point on the graph of \( f(x) \),
   then \((b, a)\) is a point on the graph of \( f^{-1}(x) \).
2. Domain of \( f(x) \) = Range of \( f^{-1}(x) \)
   Range of \( f(x) \) = Domain of \( f^{-1}(x) \)
3. The graph of \( f^{-1}(x) \) is the reflection of \( f(x) \)
   over the line \( y = x \).

Example: #60)
Write an equation for the inverse function in the form \( y = f^{-1}(x) \) where
\[
f(x) = -x^2 - 2
\]

Graph \( f \) and its inverse on the same axes and give their domains and ranges.

Solution:
We take the usual steps:
\[
\begin{align*}
y &= -x^2 - 2 \\
x &= -y^2 - 2 \\
x + 2 &= -y^2 \\
-x - 2 &= y \\
\sqrt{-x - 2} &= y \\
f^{-1}(x) &= \sqrt{-x - 2}
\end{align*}
\]
We then graph $f$ (blue) and its inverse (red) on the same axes:

Again, we have graphed the line $y = x$, though this is not necessary. Again, though, the function and its inverse are reflections of each other across this line.

Now, since $f$ is a polynomial, the domain of $f$ is $(-\infty, \infty)$. We can tell from its graph that it has range $(-\infty, 0)$. By using Fact #2 above, the domain and range of its inverse are also $(-\infty, 0)$.

**Example: #66**

Write an equation for the inverse function in the form $y = f^{-1}(x)$ where

$$f(x) = \frac{1}{x + 2}$$

Graph $f$ and its inverse on the same axes and give their domains and ranges.

Solution:

We interchange $x$ and $y$:

$$y = \frac{1}{x + 2}$$

$$x = \frac{1}{y}$$

We can then take the reciprocal of both sides of the equation:

$$\frac{1}{x} - y + 2$$

$$\frac{1}{x} - 2 = y$$

Then

$$f^{-1}(x) = \frac{1}{x} - 2 = \frac{1 - 2x}{x}$$

We then graph $f$ (blue) and its inverse (red) on the same axes:

Now, to find the domain of $f$, we note that we are in trouble of dividing by 0 since we are dealing with a fraction. Therefore, we must exclude all values which make the denominator 0, therefore, the domain of $f$ is all real numbers other than $-2$, or

domain of $f = (-\infty, -2) \cup (-2, \infty)$

The domain of the inverse is similar – we must not divide by 0,

domain of $f^{-1} = (-\infty, 0) \cup (0, \infty)$

By Fact #2 above,
range of $f = \{ -\infty, 0 \} \cup \{ 0, \infty \}$

range of $f^{-1} = \{ -\infty, -2 \} \cup \{ -2, \infty \}$

Example: #35

Decide whether the functions are inverses where $f$ and $g$ are given by the following charts:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>-4</td>
</tr>
<tr>
<td>2</td>
<td>-6</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>1</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>$g(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>3</td>
</tr>
<tr>
<td>-6</td>
<td>2</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

Remember that we want

$[g \circ f](x) = x$

$[g \circ f](x) = x$

For instance, if $x = 3$, we have

$[g \circ f](3) = g(f(3))$

$= g(-4)$

$= 3$

We do not have to check every entry in the table, however. If we switch the columns in one table and it matches the other table, then they are inverses. In this case, they are, so $f$ is the inverse of $g$ and $g$ is the inverse of $f$.

Example: #51

If the function is one-to-one, find its inverse.

$f = \{ (-3, 8), (2, 1), (5, 9) \}$

Solution:

Every $x$-value is paired with exactly one $y$-value, so the function is one-to-one and has an inverse. To find it, we just swap the $x$'s and $y$'s. Therefore,

$f^{-1} = \{ (8, -3), (1, 2), (9, 5) \}$

Example: #83

Suppose $f(x)$ is the number of cars that can be built for $x$ dollars. What does $f^{-1}(1000)$ represent?

Solution:

We note that $f(x)$ turns $x$ dollars into a certain number of cars. The inverse should "undo" this; in other words, $f^{-1}(x)$ takes cars and gives back dollars. $f^{-1}(1000)$ is the $\$$ amount it takes to build 1000 cars.
Exponential Functions, 4.2

Let $a$, $b$, and $m$, $n$ be some real numbers. We have the following useful exponential rules:

Product Rule: $a^m a^n = a^{m+n}$

Power Rule #1: $(a^m)^n = a^{mn}$

Power Rule #2: $(a b)^m = a^m b^m$

Power Rule #3 (b ≠ 0): $(\frac{a}{b})^n = \frac{a^n}{b^n}$

Zero Exponent Rule (a ≠ 0): $a^0 = 1$

If a is any positive real number not equal to 1, then:

$a^x$ is a unique real number for all real numbers $x$

$a^x = a^y$ if and only if $x = y$

If $a > 1$ and $m < n$, then $a^m < a^n$. For example:

$2^0 < 2^1 < 2^2 < 2^3 < 2^4 \ldots$

$1 < 2 < 4 < 8 < 16 \ldots$

(In other words, larger exponents of large numbers lead to larger numbers.)

If $0 < a < 1$ and $m < n$, then $a^m > a^n$. For example:

$\frac{1}{2} > \frac{1}{3} > \frac{1}{4} > \frac{1}{5} > \frac{1}{6} \ldots$

$1 > \frac{3}{2} > \frac{4}{3} > \frac{5}{4} > \frac{6}{5} \ldots$

(In other words, larger exponents of small numbers lead to smaller numbers.)

Euler’s number: $e$

Above, we said that $a$ and $b$ could be almost any real number. One number in particular, Euler’s number $e$, shows up often. To nine decimal places

$e = 2.718281828\ldots$

$e$ is not a variable, but a constant, like $\pi = 3.14152\ldots$

Example: #52)

Solve the equation

$\left(\frac{2}{3}\right)^x = \frac{9}{4}$

Solution:

We begin by noticing that $\frac{9}{4}$ may be written as

$\frac{9}{4} = \left(\frac{3}{2}\right)^2 = \left(\frac{2}{3}\right)^{-2}$

so that the original equation can be written

$\left(\frac{2}{3}\right)^x = \left(\frac{2}{3}\right)^{-2}$

Now, since the bases are equal, the exponents must be equal. Therefore, $x = -2$

Example: #50)

Solve the equation

$125^r = 5$

Solution:

We begin by noting that 125 can be written as $5^3$. Therefore, the equation can be written as

$125^r = 5$

$\left(5^3\right)^r = 5^1$

$5^{3r} = 5^1$

Now, since the bases are equal, the exponents must be equal.

$3r = 1$ so that $r = \frac{1}{3}$
Example: \#56)

Solve the equation
\[ e^{3-x} = (e^3)^{-x} \]

Solution:

We use the above properties to conclude:
\[ e^{3-x} = [e^3]^{-x} \]
\[ e^{3-x} = e^{-3x} \]

Since the bases are equal, the exponents must be equal.
\[ 3 - x = -3x \]
\[ 3 = 2x \]
\[ x = -\frac{3}{2} \]

Example: \#67)

Solve the equation
\[ r^{\frac{3}{2}} = 4 \]

Solution:

We take the 3/2 power of both sides of the equation:
\[ \left( r^{\frac{3}{2}} \right)^{\frac{2}{3}} = (4)^{\frac{2}{3}} \]
\[ r^{\frac{3}{2} \cdot \frac{2}{3}} = (\sqrt{4})^3 \]
\[ r^1 = (2)^3 \]
\[ r = 8 \]

Definition:

An exponential function with base \( a \) is defined by \( f(x) = a^x \) where \( a > 0 \) and \( a \neq 1 \).

Like any function, we can graph \( f(x) = a^x \). The graph:

- Always contains the points \((-1, \frac{1}{a}), (0, 1), (1, a)\).
- If \( a > 1 \), then \( f \) is an increasing function.
- If \( 0 < a < 1 \), then \( f \) is a decreasing function.
- The \( x \)-axis is a horizontal asymptote.
- Domain = \((-\infty, \infty)\)
- Range = \([0, \infty)\)

Example: \#13)

Graph the function
\[ f(x) = 3^x \]

Solution:

Since \( a = 3 \), we know that the graph contains the following points:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>1/3</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

Since \( a > 1 \), this is an increasing function. The \( x \)-axis is a horizontal asymptote. The graph never reaches the \( x \)-axis, so that the \( y \)-value approaches zero, but never reaches it.
Example: #15)

Graph \( f(x) = \left(\frac{1}{3}\right)^x \)

Solution:

Here we have \( a = \frac{1}{3} \), so we are expecting a decreasing graph. We have

\[
\begin{array}{c|c}
 x & y \\
-1 & 3 \\
0 & 1 \\
1 & 1/3 \\
\end{array}
\]

Notice that this is a reflection of the previous graph over the y-axis. You can use the techniques of Chapter 2 to understand this reflection.

Example: #34)

Graph \( f(x) = \left(\frac{1}{3}\right)^x + 4 \)

Solution:

We note that this is just the previous graph, translated up by four units.

\[
\begin{array}{c|c}
 x & y \\
-1 & 7 \\
0 & 5 \\
1 & \frac{11}{3} + 4.3 \\
\end{array}
\]

Note that the range is now \((4, \infty)\).

**Compound interest**

is money that is compounded (or paid) a number of times per year. If the formula is compounded a finite number of times per year, we use the following formula

\[\begin{align*}
A &= P \left(1 + \frac{r}{n}\right)^{nt} \\
\end{align*}\]

where:

- \( A \) = future value
- \( P \) = present value
- \( r \) = annual interest rate (as a decimal – 5% is .05, not 5)
- \( n \) = number of times per year interest is compounded
- \( t \) = time in years

**Example:**

How much money will be in an account if you deposit $1000 at 6% interest compounded annually for 3 years?

Solution:

\[
\begin{align*}
P &= 1000 \\
r &= 0.06 \\
n &= 1 \text{ ("compounded annually" means one time per year)} \\
t &= 3 \\
\end{align*}\]

Then we calculate

\[
A = 1000 \left(1 + \frac{0.06}{1}\right)^{1(3)} = 1000(1.06)^3 = 1191.02
\]

Note that we do not round until the last step, and we usually round to the nearest penny.

**Continuous compounding:**
We can think of situations in which we compound once per year, then 10 times, then 100 times, etc. We can, in fact, think of what would occur if we compounded continuously. If this is the case, we use the formula

\[ A = Pe^{rt} \]

where

- \( A \) = future value
- \( P \) = present value
- \( r \) = annual interest rate (as a decimal)
- \( t \) = time (years)

**Example: #72**

Find the future value and interest earned if $56,780 is invested at 5.3% compounded quarterly for 23 quarters.

Solution:

We calculate

\[ A = 56780 \left( 1 + \frac{0.053}{4} \right)^{23 	imes 4} \]

\[ = 56780 \times (1.013125)^{23} \]

\[ \approx 76,855.95 \]

Find the future value and interest earned if $60,780 is invested at 5.3% compounded continuously for 16 years.

Solution:

We calculate

\[ A = 56780e^{0.053 \times 16} \]

\[ = 56780e^{0.8488} \]

\[ \approx 72,535.96 \]

**Example: #74**

Find the present value of $45,000 if interest is 3.6% compounded monthly for 1 yr.

Solution:

We identify the quantities:

- \( A \) = $45,000
- \( r = 0.036 \)
- \( n = 12 \) times a year
- \( t = 1 \) year

and set up the equation

\[ 45000 = P \left( 1 + \frac{0.036}{12} \right)^{1 \times 12} \]

\[ 45000 = P \times (1.003)^{12} \]

\[ 45000 \times \left( \frac{1}{1.033^{12}} \right) = P \]

\[ P = 43,411.15 \]

**Example: #78**

Find the required annual interest rate to the nearest tenth of a percent for $5000 to grow to $8400 if interest is compounded quarterly for 8 years.

Solution:

We identify the known quantities and set up an equation:

- \( P = $5,000 \)
- \( A = $8,400 \)
- \( n = 4 \) times a year
- \( t = 8 \) years

\[ 8400 = 5000 \left( \frac{1 + \frac{r}{4}}{4} \right)^{8 \times 4} \]

\[ \frac{8400}{5000} = \left( \frac{1 + \frac{r}{4}}{4} \right)^{32} \]
\[
\frac{\sqrt{8400}}{5000} = \left(1 + \frac{r}{4}\right)
\]
\[
\frac{\sqrt{8400}}{5000} - 1 = \frac{r}{4}
\]
\[
r = 4 \left(\frac{\sqrt{8400}}{5000} - 1\right)
\]

\[r \approx 0.065\]

The annual interest rate must be 6.5%.

**Example: #84**

A person learning certain skills involving repetition tends to learn quickly at first. Then learning tapers off and approaches some upper limit. Suppose the number of symbols per minute that a person using a word processor can type is given by

\[p(t) = 250 - 120(2.8)^{-0.22t}\]

where \(t\) is the number of months the operator has been in training.

Find each value:

\[p(2) = 250 - 120(2.8)^{-0.22(2)} \approx 207.14\]
\[p(4) = 250 - 120(2.8)^{-0.22(4)} \approx 234.69\]
\[p(10) = 250 - 120(2.8)^{-0.22(10)} \approx 249.30\]

After a while, we note that the value of \(p\) is getting closer to 250 symbols per minute as \(t\), the number of months spent in training, gets larger.
Textbook Section 4.3
Logarithmic Functions

Objectives

- The student will be able to convert between logarithmic and exponential forms
- The student will be able to solve logarithmic equations
- The student will be able to graph basic logarithmic functions
- The student will be able to graph logarithmic functions using transformations
- The student will be able to write an equation for the given graph
- The student will be able to use properties of logarithms to write logarithms in expanded form
- The student will be able to use properties of logarithms to combine expressions as a single logarithm
- The student will be able to evaluate logarithms

Key Concepts

The inverse of an exponential function is a logarithmic function.

Logarithm:

\[ x = a^y \text{ if and only if } y = \log_a x \]

for all real numbers \( y \) and all positive numbers \( a \) and \( x \), where \( a \neq 1 \).

- A logarithm is an exponent
- The expression represents the exponent to which the base \( a \) must be raised in order to obtain \( x \)

<table>
<thead>
<tr>
<th>Exponential Form</th>
<th>Logarithmic Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = a^y )</td>
<td>( y = \log_a x )</td>
</tr>
</tbody>
</table>

argument = base\(^{\text{exponent}}\) \hspace{2cm} \text{exponent} = \log_{\text{base}}(\text{argument})

**NOTE:** Read "log base \( a \) of \( x \)"

Logarithms solve for the exponent!!

**NOTE:** \( \log_a(\text{negative number}) \) is undefined

Logarithm Equations:

- Convert between exponential and logarithmic forms to solve.
- Use properties of exponents as needed.

Properties of Logarithms:

For \( x > 0 \), \( y > 0 \), \( a > 0 \), \( a \neq 1 \), and any real number \( r \),

1. \( \log_a(1) = 0 \)
2. \( \log_a(a) = 1 \)
3. \( \log_a(a^r) = r \)
4. \( a^{\log_a(x)} = x \)
5. Product Property \( \log_a(xy) = \log_a(x) + \log_a(y) \)
6. Quotient Property \( \log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y) \)
7. Power Property \( \log_a(x^r) = r \log_a(x) \)

Definition:

A logarithmic function with base \( a \) is defined by \( f(x) = \log_a(x) \) where \( a > 0 \), \( a \neq 1 \), and \( x > 0 \)
Characteristics of logarithmic graphs:

\[ f(x) = \log_a(x) \]

1. Always contains the points \( \left( \frac{3}{2}, -1 \right), (1, 0) \) and \((a, 1)\)

2. If \( a > 1 \), \( f \) is an increasing function.
   If \( 0 < a < 1 \), \( f \) is a decreasing function.

3. The y-axis is a vertical asymptote.
4. Domain = \((0, \infty)\)
   Range = \((-\infty, \infty)\)

Textbook Section 4.4
Evaluating Logarithms and Change-of-Base Theorem

Objectives

- The student will be able to use the change-of-base theorem to evaluate a logarithm
- The student will be able to solve application problems

Key Concepts

Common Logarithm:

\[ \log_{10}(x) = \log(x) \]

Natural Logarithm:

\[ \log_e(x) = \ln x \]

Applications and Modeling:

\( \text{pH problems:} \)

\( \text{pH} = -\log[H_2O^+] \)

Measuring the loudness of sound

\( d = 10\log_{10}\left(\frac{I}{I_0}\right) \)

Change-of-Base Theorem:

\[ \log_a(x) = \frac{\log_b(x)}{\log_b(a)} \]

for any positive real numbers \( x, a, \) and \( b, \) where \( a \neq 1 \) and \( b \neq 1 \)

Rewrite a log with a new base.

- \( a = \) old base
- \( b = \) new base

Uses for the change-of-base theorem:

- Evaluate a log that is not base 10 or \( e \)
- Solve an equation

Online Homework and Quiz Assignments

After reviewing the Key Concepts, log into MyLabsPlus and begin your homework and quiz for this week. Then go to www.ucty.mylabsplus.com and begin working on your assignments.
Logarithmic Functions, 4.3

We note that exponential functions are one-to-one, so they should have an inverse. The inverse is the logarithmic function. Normally, we would try:

\[ y = a^x \]

We interchange \( x \) and \( y \)

\[ x = a^y \]

and solve for \( y \) ... but we can't. We need a new function.

**Logarithm:** We say that

\[ x = a^y \] (exponential form)

if and only if

\[ y = \log_a x \] (logarithmic form)

for all real numbers \( y \) and all positive numbers \( a \neq 1 \) and \( x \). Logs of negative numbers are undefined. We have some examples:

<table>
<thead>
<tr>
<th>Exponential Expression</th>
<th>Logarithm Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( 3^2 = 9 )</td>
<td>( \log_3 9 = 2 )</td>
</tr>
<tr>
<td>2. ( 4^2 = 16 )</td>
<td>( \log_4 16 = 2 )</td>
</tr>
<tr>
<td>3. ( e^2 = 9 )</td>
<td>( \log_e 9 = x )</td>
</tr>
<tr>
<td>4. ( a^2 = 4 )</td>
<td>( \log_a 4 = 5 )</td>
</tr>
</tbody>
</table>

**Example: #13**

Solve \( x = \log_5 \frac{1}{5} \)

Solution: We convert this into exponential form by noting that

- base = 5
- exponent = \( x \)
- argument = \( \frac{1}{5} \)

Then

\[ 5^x = \frac{1}{5} \]

\[ 5^x = 5^{-1} \]

\[ 5^x = (5^{-1})^{-1} \]

\[ 5^x = 5^4 \]

Now, since the bases are the same, the exponents must be the same, so that \( x = 4 \).

**Example: #17**

Solve \( x = \log_9 \sqrt{9} \)

Solution: We convert this into exponential form:

\[ 9^x = \frac{1}{9} \]

\[ 9^x = 9^{-1} \]

Now, since the bases are the same, the exponents must be the same.

Therefore, \( x = \frac{1}{2} \).

---

**Properties of Logarithms:**

For \( x > 0, y > 0, a > 0, a \neq 1 \), and any real number \( r \),

1. **Product Property** \( \log_a xy = \log_a x + \log_a y \)

2. **Quotient Property** \( \log_a \frac{x}{y} = \log_a x - \log_a y \)
3. Power Property \( \log_b x^n = n \log_b x \n\)

4. Log of base \( \log_b b = 1 \)

5. Log of 1 \( \log_b 1 = 0 \)

6. Inverse Property \#1 \( \log_b a^x = x \)

7. Inverse Property \#2 \( a^{\log_b x} = x \)

**Example: \#61**

Use the properties of logarithms to rewrite the expression \( \log_b \sqrt[3]{7} \). Simplify the result if possible.

**Solution:** We write

\[
\log_b \sqrt[3]{7} = \log_b 7^{1/3} - \log_b 3
\]

\[
= \log_b 7^{1/3} - \log_b 3
\]

\[
= \log_b 7^{1/3} - \log_b 3
\]

At this point, the only further simplification would be to substitute actual numerical values. However, we will often like to keep it in this form.

**Example: \#63**

Simplify \( \log_b (2x + 5y) \)

**Solution:** This is already simplified. A common error is to write

\[
\log_b (2x + 5y) = \log_b 2x + \log_b 5y
\]

but this is NOT one of the properties of the logarithm and all calculations of this sort are incorrect.

**Example:**

Given that \( \log_{10} 2 = .3010 \), \( \log_{10} 3 = .4771 \) calculate \( \log_{10} 6 \)

**Solution:** We know that

\[
\log_{10} 6 = \log_{10} (2 \cdot 3)
\]

\[
= \log_{10} 2 + \log_{10} 3
\]

\[
= .3010 + .4771 = .7781
\]

**Example: \#66**

Use the properties of logarithms to rewrite

\[
\log_b \left( \frac{m^n x^4}{z^2} \right)
\]

Simplify the result if possible.

**Solution:** We write

\[
\log_b \left( \frac{m^n x^4}{z^2} \right) = \log_b \left( m^n x^4 \right) - \log_b (z^2)
\]

\[
= \frac{1}{3} \log_b m^n - \frac{1}{3} \log_b x^4 - \frac{1}{3} \log_b (z^2)
\]

\[
= \frac{1}{3} (5 \log_b m + 4 \log_b x - 2 \log_b z)
\]
$$\frac{5}{3} \log_{10} m + \frac{4}{3} \log_{10} n = \frac{2}{3} \log_{10} r$$

**Example: #73**

Write $\log_{10} m - \log_{10} n + \log_{10} r$ as a single logarithm with coefficient 1.

**Solution:** We write

$$\log_{10} m - \log_{10} n + \log_{10} r = \log_{10} \frac{m}{n} + \log_{10} r$$

$$= \log_{10} \frac{mr}{n}$$

**Definition:** A logarithmic function with base $a$ is defined by

$$f(x) = \log_{10} x$$

where $a > 0$, $a \neq 1$, and $x > 0$.

**Example: #45**

Graph $y = 5^x$ and its inverse function $y = \log_{10} x$.

**Solution:** We already know how to graph $y = 5^x$, which is pictured below in red. All we need to do to graph the logarithm, in black, is flip over the line $y = x$.

To get the test points, we can simply flip the test points from the graph of the exponential, since these are inverses. In the same way, we can figure out the domains and ranges:

<table>
<thead>
<tr>
<th>$y = 5^x$</th>
<th>$y = \log_{10} x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$y$</td>
</tr>
<tr>
<td>-1</td>
<td>-0.3</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>-1</td>
<td>5^{-1} = 0.2</td>
</tr>
</tbody>
</table>

**Characteristics of logarithmic graphs:**

If we consider the graph of $f(x) = \log_{10} x$ then we know that

1. Always contains the points $(1, 0), (0, 0), (0, 1)$
2. If $a > 1$, $f$ is an increasing function.
   - If $0 < a < 1$, $f$ is a decreasing function.
3. The y-axis is a vertical asymptote.
4. Domain = $(0, \infty)$
   - Range = $(-\infty, \infty)$
Example: #33)

Graph \( \log_{2}x + 3 \)

Solution: We start with the basic graph of \( \log_{2}x \), which we know contains the points

\[
\left( \frac{1}{4}, -1 \right), \left( 1, 0 \right), \left( 2, 1 \right)
\]

and shift these up 3 units by adding 3 to the \( y \)-values:

\[
\left( \frac{1}{4}, 2 \right), \left( 1, 3 \right), \left( 2, 4 \right)
\]

Therefore, we have the following graph:

![Graph of \( \log_{2}x + 3 \)]

Example: #34)

Graph \( \log_{2}(x + 3) \)

Solution: This is a graph of the same basic function, but to the left three units.

Note that the domain is \((-3, \infty)\).
Evaluating Logarithms and Change-of-Base Theorem, 4.4

Common Abbreviations: Remember that there are an infinite number of logarithmic functions of the form

\[ f(x) = \log_{a}x \]

where \( a \), the base, is any number greater than 0, not equal to 1. There are two bases in particular that you will come across most frequently in applications.

When we talk about the common logarithm or common log, this means we are using base 10. For all positive numbers \( x \), we use the shorthand

\[ \log_{10}x = \log x \]

In other words, if the base is omitted on a log, the base is 10.

The natural logarithm uses base \( e \), where

\[ e = 2.718281828459 \ldots \]

is a number, not a variable. For all positive numbers \( x \), we have the shorthand

\[ \log_{e}x = \ln x \]

Applications and Modeling:

The pH, a measure of acidity commonly used in chemistry, is defined as

\[ pH = -\log[H_{3}O^{+}] \]

where \([H_{3}O^{+}]\) is the hydronium ion concentration in moles per liter.

Example: #29)

A grapefruit has \([H_{3}O^{+}] = 6.3 \times 10^{-4}\).

What is its pH?

Solution:

\[ pH = -\log[H_{3}O^{+}] = -\log[6.3 \times 10^{-4}] \]

If you are using a TI-30Xa, you can calculate this number as follows:

1. 6.3
2. [EE button]
3. 4
4. [=]
5. [log button]
6. [=] (negative sign)
7. [=]

= 2.1997

Alternatively, we can do more of the calculation on paper:

\[ pH = -\log[H_{3}O^{+}] = -\log(6.3 \times 10^{-4}) \]
\[ = -\log(6.3) - \log(10^{-4}) \]
\[ = -\log(6.3) + (-4)\log(10) \]
\[ = -\log(6.3) + (-4)(1) \]
\[ = -\log 6.3 + 4 \]
\[ = -0.79934 + 4 \]
\[ = 3.2007 \]

Example: #29)

Soda pop has a pH of 2.7.

What is its \([H_{3}O^{+}]\)?

Solution:

\[ pH = 2.7 = -\log[H_{3}O^{+}] \]

\[-2.7 = \log[H_{3}O^{+}] \]

Then we convert to exponential form:

Base = 10
Exponent = -2.7
\[ x = [H_{3}O^{+}] \]
\[ x = [H_{3}O^{+}] = 10^{-2.7} \]
We can use the calculator to find:

1. 10
2. \( \sqrt{7} \)
3. 2.7
4. (negative sign)
5. [-]

which should yield \( [H_3O^+] = \text{ or } = 0.01995262... \)

You may convert this to scientific notation by pressing \( [\text{EE}] [5] \).

**Example: #37)**

Suppose that water from a wetland area is sampled and found to have the given hydronium ion concentration

\[ [H_3O^+] = 6.3 \times 10^{-4} \]

Determine whether the wetland is a rich fen (pH is 6.0-7.5), poor fen (4.0-6.0), or bog (3.0 or less).

Solution:

\[ pH = -\log([H_3O^+]) = -\log(6.3 \times 10^{-4}) = 4.6038 \]

Since the pH is between 4.0 and 6.0, it is classified as a poor fen.

**Example: #46)**

To measure the loudness of sound, we define the decibel rating \( d \), where

\[ d = 10 \log \frac{l}{l_0} \]

\( d \) = decibel rating

\( l \) = intensity of sound being measured

\( l_0 \) = intensity of very faint sound (threshold sound)

Rock music has intensity of 895,000,000,000 \( l_0 \). What is its decibel rating?

Solution:

\[ d = 10 \log \left( \frac{895 \times 10^{11}}{l_0} \right) = 10 \log \left( 8.95 \times 10^{11} \right) = 10 \log (8.95 \times 10^{11}) \]

We can use the calculator to find this:

1. 65.5
2. \([\text{EE}]\)
3. 11
4. \([\log]\)
5. \([*] 10 [-]\)

which should yield \( d = 119.518 \)

so rock music has a decibel rating of about 120.

**Example: #47)**

The magnitude of an earthquake, measured on the Richter scale, is

\[ M = \log \frac{l}{l_0} \]

where \( l \) is the amplitude registered on a seismograph 100 km from the epicenter of the earthquake and \( l_0 \) is the amplitude of an earthquake of a certain (small) size. Find the Richter scale ratings for earthquakes having the amplitude 1,000,000 \( l_0 \).

Solution:

\[ M = \log \frac{l}{l_0} = \log \left( \frac{10000000}{l_0} \right) = \log 1000000 = 6 \]

The earthquake has a magnitude of 6 on the Richter scale.
Example: #33)
The number of species in a sample is given by
\[
S(n) = an \ln \left(1 + \frac{n}{a}\right)
\]
where
- \(n\) is the number of individuals in the sample
- \(a\) is a constant that indicates the diversity of species in the community
If \(a = 0.36\), find \(S(60)\) for \(n = 200\).

Solution:
We calculate
\[
S(200) = 0.36 \ln \left(1 + \frac{200}{0.36}\right)
\]
\[
= 0.36 \ln (555.5555)
\]
If you are using a TI-30XI, you can calculate this by
1. 200 [+] 1 36
2. [+] 1
3. [ln]
4. [+] 36 [→]
which should yield
\[
S(200) = 2.2758...
\]
We round at only the last step, to conclude that there are
\[
S(200) = 2
\]
species in a sample of 200 individuals.

********

Change-of-Base Theorem:
Again, there are an infinite number of logarithmic functions of the form
\[
f(x) = \log_a x
\]
where \(a\), the base, is any number greater than 0, not equal to 1. How do we calculate a logarithm if the base isn't \(e\) or 10? We don't have a button for base 2, for instance, on our calculators. We have the change-of-base theorem:
\[
\log_a x = \frac{\log_b x}{\log_b a}
\]
for any positive real numbers \(x\), \(a\), and \(b\), where \(a, b \neq 1\). We can rephrase a logarithm in base \(a\) in terms of base \(b\), and we can choose any base \(b\).

Example:
Calculate \(\log_2 5\) to four decimal places.

Solution:
We do not have a log base 2 button on our calculators, so we cannot calculate this directly. Using the change-of-base theorem, however, we can use base \(e\) or 10 as follows:
\[
\log_2 5 = \frac{\ln 5}{\ln 2}
\]
\[
= \frac{1.609437912}{0.693147181}
\]
\[
= 2.3219
\]
Note that the intermediate calculations are different, but the final answers are the same. This will always be the case if you've done the calculations correctly. The base you choose doesn't matter, as long as they are consistent.

Example: #63)
Calculate \(\log_{10} 0.59\) to four decimal places.
\[
\log_{10} 0.59 = \frac{\ln 0.59}{\ln 10}
\]
\[
= -0.2537
\]

Solving Equations:
We can use logarithms to solve equations that would otherwise be difficult (if not impossible) to solve otherwise.
Example: 4.5.6)

Solve $5^x = 13$

Solution:

We write the equation in logarithmic form:

$$\log_5(13) = x$$

and use the change-of-base theorem:

$$x = \frac{\log_{10}13}{\log_{10}5} = \frac{\ln 13}{\ln 5} \approx 1.594$$

where we've rounded to the nearest thousandth.

Example: 75.

Let $u = \ln a$, $v = \ln b$. Write

$$\ln \left( \frac{a^2}{b^2} \right)$$

in terms of $u$ and $v$, without using the ln function.

Solution:

Since the square root can be represented with a power of ½, we write

$$\ln \left( \sqrt{\frac{a^2}{b^2}} \right) = \ln \left( \left( \frac{a}{b} \right)^2 \right)^{1/2} = \frac{1}{2} \ln \left( \frac{a}{b} \right)^2$$

where in this last step we've used Log Property #3 to bring the exponent down. Using Log Property #2,

$$\frac{1}{2} \ln \left( \frac{a}{b} \right)^2 = \frac{1}{2} \left( \ln a^2 - \ln b^2 \right)$$

Using Log Property #3 again, we bring down the exponents and then distribute:

$$\frac{1}{2} (\ln a - \ln b) = \frac{1}{2} \ln a - \frac{1}{2} \ln b$$

Then we substitute $u = \ln a$, $v = \ln b$ to obtain

$$\frac{3}{2} \ln a - \frac{3}{2} \ln b = \frac{3}{2} u - \frac{3}{2} v$$

Example: 79.a)

Given $f(x) = \ln x$ evaluate

a. $f(e^4)$

Solution:

$$f(e^4) = \ln e^4 = 4 \ln e = 4(1) = 4$$

Example: 79.b)

Given $f(x) = \ln x$ evaluate

b. $f(e^{3\ln a})$

Solution:

$$f(e^{3\ln a}) = 3 \ln a \ln e = 3 \ln a$$

= $2 \ln 3$
Textbook Section 4.5
Exponential and Logarithmic Equations

Objectives

- The student will be able to solve exponential equations
- The student will be able to solve logarithmic equations
- The student will be able to solve an equation for a given variable
- The student will be able to solve application problems

Key Concepts

Property of Logarithms:

If $x > 0$, $y > 0$, $a > 0$, and $a \neq 1$, then $x = y$ if and only if $\log_a x = \log_a y$

Strategies for solving:

- Isolate the exponential or logarithm expression.
- If $a^{f(x)} = b$
  - solve by applying a logarithm on both sides, or
  - solve by converting to log form and using the change-of-base theorem
- If $\log_a f(x) = b$
  - solve by converting to exponential form $a^b = f(x)$
- If $\log_a f(x) = \log_a g(x)$
  - for same base $a$, $f(x) = g(x)$
- Check that the proposed solution is in the domain.
  - For above: $a$ and $b$ are real numbers with $a > 0$, and $a \neq 1$
  - Be sure that the argument of the log is positive.
  - Remember: $\log_a$ (negative number) is undefined.

Textbook Section 4.6
Applications and Models of Exponential Growth and Decay

Objectives

- The student will be able to find an exponential function that models the given data set
- The student will be able to find the doubling time and half-life
- The student will be able to solve application problems

Key Concepts

Exponential Growth or Decay Function:

$$y = y_0 e^{kt}$$

Let $y_0$ be the starting amount at time $t = 0$.
For $k > 0$, this models exponential growth.
For $k < 0$, this models exponential decay.

Half-Life

the time it takes for a decaying substance to become half of its initial amount

Newton’s Law of Cooling:

Describes the rate at which an object cools $f(t) = T_0 + Ce^{-kt}$
where $C$ and $k$ are constants, $T(0)$ is the temperature of the object at time $t$, and $T_e$ is the temperature of the environment.

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**Online Homework and Quiz Assignments**

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Property of Logarithms:

If \( x > 0, \ y > 0, \ a > 0 \) and \( a \neq 1 \), then

\[ x = y \text{ if and only if } \log_a(x) = \log_a(y) \]

i.e. by taking the logarithm of both sides of an equation, we have another way to solve.

Example: \#6)

Solve \( 5^x = 13 \). Express irrational solutions as decimals correct to the nearest thousandth.

Solution: We take the log of both sides:

\[
\log(5^x) = \log(13)
\]

\[x \log(5) = \log(13)\]

\[x = \frac{\log(13)}{\log(5)}\]

\[x \approx 1.594\]

Example: \#11)

Solve \( 4^{-x-1} = 3^{2x} \)

Solution: We begin by taking the natural log of both sides:

\[
\ln(4^{-x-1}) = \ln(3^{2x})
\]

\[(x+1)\ln(4) = (2x)\ln(3)\]

\[x\ln(4) - \ln(4) = 2x\ln(3)\]

\[x\ln(4) - 2x\ln(3) = \ln(4)\]

\[x(\ln(4) - 2\ln(3)) = \ln(4)\]

\[x = \frac{\ln(4)}{\ln(4) - 2\ln(3)} \approx -1.710\]

Example: \#13)

Solve \( 6^{x+1} = 4^{2x-1} \)

Solution: We begin by taking the common log of both sides:

\[
\log(6^{x+1}) = \log(4^{2x-1})
\]

\[(x+1)\log(6) = (2x-1)\log(4)\]

\[x\log(6) + \log(6) = 2x\log(4) - \log(4)\]

\[x\log(6) - 2x\log(4) = -\log(4) - \log(6)\]

\[x(\log(6) - 2\log(4)) = -\log(4) - \log(6)\]

\[x = \frac{-\log(4) - \log(6)}{\log(6) - 2\log(4)}\]
Example: #24)

Solve $5(1.2)^{3x-2} + 1 = 7$

Solution:

$5(1.2)^{3x-2} + 1 = 7$

$5(1.2)^{3x-2} = 6$

$(1.2)^{3x-2} = \frac{6}{5} = 1.2$

$(1.2)^{3x-2} = (1.2)^1$

Now, since the bases are the same, the exponents must be equal:

$3x - 2 = 1$

$3x = 3$

$x = 1$

Strategies for solving:

1. Isolate the exponential or logarithm expression.
2. Identify the form:
   - If $a^{f(x)} = b$ you may solve by applying a logarithm on both sides or solve by converting to log form and using the change-of-base theorem.
   - If $\log_a f(x) = b$, solve by converting to exponential form $a^b = f(x)$.
   - If $\log_a f(x) = \log_a g(x)$, since these have the same base, solve $f(x) = g(x)$.
3. Check that the proposed solution is in the domain.

Example: #29)

Solve $5 \ln x = 10$

Solution: We isolate the log and convert to exponential form:

\[
\frac{\ln x}{5} = \frac{10}{5}
\]

\[
\ln x = 2
\]

\[
x = e^2
\]

Example: #23)

Solve $\log_6 (2x + 4) = 2$

Solution: We convert to exponential form:

\[
2x + 4 = 6^2 = 36
\]

\[
2x = 32
\]

\[
x = 16
\]

Next we check. Most important is to remember that the log of a negative number is undefined. We plug in 16 for $x$:

\[
\log_6 (2(16) + 4) = \log_6 36 = 2
\]

So the solution $x = 16$ is correct.

Example: #41)
Solve \( \log x + \log(x - 21) = 2 \)

Solution: We combine the left-hand side into a single log:
\[
\log[x(x - 21)] = 2
\]
\[
\log[x^2 - 21x] = 2
\]

Then we remember that this the common log, base 10, and convert to exponential form:
\[x^2 - 21x - 10^2 = 100\]

This is just a quadratic equation, which we can solve easily by any of our usual methods to obtain:
\[x = -4.25\]

However, we've not done – we need to check the answers. We first try \( x = 25 \):
\[\log(25) + \log(25 - 21) = 2\]

So \( x = 25 \) is a valid solution. However, when we try \( x = -4 \), we obtain
\[\log(-4) + \log(-4 - 21)\]

which is undefined. \( x = 4 \) is not a solution. The final answer, then, is \( x = 25 \).

---

**Example: #46**

Solve \( \ln(5 + 4x) - \ln(3 + x) = \ln 3 \)

Solution: We begin by using the properties of logs to combine the two logs on the left-hand side:
\[\frac{5 + 4x}{3 + x} = \ln 3\]

Now, since both logarithms have the same base, we can simply get rid of the logs and solve:
\[\frac{5 + 4x}{3 + x} = 3\]
\[5 + 4x = 3(3 + x) = 9 + 3x\]
\[5 + x = 9\]
\[x = 4\]

As usual, we need to plug this \( x \) back into the original equation to make sure that all logs are positive, they are. In this case, \( x = 4 \) is a valid solution.

---

**Example: #54**

Solve \( \log x = \sqrt{\log x} \)

Solution: We begin by squaring both sides and solving for \( \log x \):
\[(\log x)^2 = (\sqrt{\log x})^2\]
\[(\log x)^2 = \log x\]
\[(\log x)^2 - \log x = 0\]

Thus, the factors are:
\[\log(\log x - 1) = 0\]

Now, by the zero-factor property, each of these factors can be set equal to 0 to obtain a solution:
\[\log x = 0\]
\[10^0 = x\]
\[1 = x\]

So that \( x = 1 \) is the first possible solution.
\[\log x = 1\]
\[10^1 = x\]
\[10 = x\]

So we have \( x = 10 \) also. We must plug these back in to the original equation to be sure that they are valid solutions. In fact, both work, so \( x = 1 \) and \( x = 10 \) are solutions.
Example: #63)

Solve the equation for \( t \).

\[
I = \frac{E}{R} \left( 1 - e^{-\frac{Rt}{E}} \right)
\]

Solution: First, we isolate the exponential:

\[
\frac{R}{E} I = 1 - e^{-\frac{Rt}{E}}
\]

\[
\frac{RI}{E} - 1 = -e^{-\frac{Rt}{E}}
\]

\[
1 - \frac{RI}{E} = e^{-\frac{Rt}{E}}
\]

Now we can take the logarithm (base \( e \)) of both sides:

\[
\ln\left(\frac{RI}{E} - 1\right) = \ln\left(e^{-\frac{Rt}{E}}\right)
\]

and we can solve for \( t \):

\[
t = -\frac{2}{R} \ln\left(1 - \frac{RI}{E}\right)
\]

Example: #70)

Solve the equation for \( x \).

\[ D = 160 + 10 \log x \]

Solution: We begin by isolating the logarithm:

\[ D - 160 = 10 \log x \]

\[ \frac{D - 160}{10} = \log x \]

We then convert to exponential form:

\[ x = 10^{\frac{D - 160}{10}} \]

Example: #71)

If $10,000 is invested in an account at 3% annual interest compounded quarterly, how much will be in the account in 5 years if no money is withdrawn?

Solution: Since we're compounding quarterly, four times a year, we use

\[
A = P \left(1 + \frac{r}{n}\right)^{nt}
\]

\[ A = 10,000 \left(1 + \frac{0.03}{4}\right)^{4(5)} \]

\[ A = 10,000(1.0075)^{20} \]

\[ A = 11,611.84 \]

Note that we only round in the last step!

Example: #72)

If $5000 is invested in an account at 4% annual interest compounded continuously, how much will be in the account in 8 years if no money is withdrawn?

Solution: Since we're compounding continuously, we use

\[ A = Pe^{rt} \]
\[ A = 5000e^{-4t} \]
\[ A = 50865.64 \]

Example: #80)

At the World Championship races held in Rome's Olympic Stadium in 1987, American sprinter Carl Lewis ran the 100-m race in 9.88 seconds. His speed in meters per second after \( t \) seconds is closely modeled by the function defined by

\[ f(t) = 11.65\left(1 - e^{-\frac{t}{1.27}}\right) \]

a. How fast was he running as he crossed the finish line?

Solution: He crosses the finish line at \( t = 9.88 \) sec, so we calculate

\[ f(9.88) = 11.65\left(1 - e^{-\frac{9.88}{1.27}}\right) \]

\[ f(9.88) = 11.645 \text{ m/sec} \]

b. After how many seconds was he running at the rate of 10 m per sec?

Solution: We set

\[ 10 = 11.65\left(1 - e^{-\frac{t}{1.27}}\right) \]

\[ \frac{10}{11.65} = 1 - e^{-\frac{t}{1.27}} \]

\[ \frac{10}{11.65} - 1 = -e^{-\frac{t}{1.27}} \]

\[ -1 = \frac{10}{11.65} - e^{-\frac{t}{1.27}} \]

\[ \ln\left(1 - \frac{10}{11.65}\right) = -\frac{t}{1.27} \]

\[ -1.27\ln\left(1 - \frac{10}{11.65}\right) = t \approx 2.49 \]
Applications and Models of Exponential Growth and Decay, 4.6

In modeling exponential growth or decay, we begin with the amount of whatever substance at time $t = 0$, which either grows or decays as $t$ increases. We have the model

$$y(t) = y_0 e^{kt}$$

where $y(t)$ is the amount of the substance at time $t$ and $k$ is the growth (or decay) constant. If $k > 0$, this models exponential growth. If $k < 0$, this models exponential decay. We do not always write $y(0)$. Sometimes you will see $A(0)$, which means the amount at time $t$, or simply $y$ or $A$.

Example: #5)

A sample of 500 g of radioactive lead-210 decays to polonium-210 according to the function described by

$$A(t) = 500e^{-0.032t}$$

where $t$ is time in years. Find the amount of radioactive lead remaining after 4 years.

Solution:

$$A(4) = 500e^{-0.032(4)}$$

$$A(4) = 439.93 \text{ g}$$

Find the amount of radioactive lead remaining after 8 years.

$$A(8) = 500e^{-0.032(8)}$$

$$A(8) = 387.07 \text{ g}$$

Example: #7)

The half-life is the amount of time it takes for a decaying substance to become half of its original amount. Find the half-life of radium-226, which decays according to the function defined by

$$A(t) = A_0 e^{-0.00043t}$$

Solution:

We don’t need to know exactly how much radium we have to begin with, since if we start with $A_0$, we end up with half that amount or $\frac{1}{2} A_0$. We can then solve:

$$\frac{1}{2} A_0 = A_0 e^{-0.00043t}$$

$$\frac{1}{2} = e^{-0.00043t}$$

In this last step, we have canceled $A_0$ from both sides, so again it doesn’t matter how much radium we started with. We then take the natural log of both sides:

$$\frac{1}{2} \ln 2 = -0.00043t$$

$$\frac{-1}{0.00043} \ln 2 = t$$

$$t = 1612 \text{ years}$$

The half-life of radium-226 is about 1612 years.

Example: #9)

If 12 g of a radioactive substance are present initially and 4 years later, only 6 g remain, how much of the substance will be present after 7 years?

Solution:

In this case we don’t know $k$, the decay constant, so we have to solve for it:

$$y = y_0 e^{kt}$$

$$6 = 12e^{4k}$$

$$k = \frac{1}{4} \ln \frac{1}{2} = -0.173286796$$

It is important not to round $k$! Now that we have the decay constant, we can find out how much of the material is present after 7 years:

$$y(7) = 12e^{-0.173286796(7)} = 3.568 \text{ g}$$

Therefore, after 7 years, 3.568 grams of the substance will remain.
Textbook Section 5.1
Systems of Linear Equations

Objectives

- The student will be able to solve a system of 2 equations using substitution
- The student will be able to solve a system of 2 equations using elimination
- The student will be able to match a system of equations with a graph in the xy-plane
- The student will be able to identify an inconsistent system
- The student will be able to identify a system with infinitely many solutions and write the solution with a given arbitrary variable
- The student will be able to solve a system of 3 equations containing 3 variables
- The student will be able to solve a system of 2 equations containing 3 variables by using an arbitrary variable
- The student will be able to solve an applied problem by creating a system of equations

Key Concepts

Definition

A system of equations is a set of equations.
A solution satisfies every equation in the set.

For two variables: x and y

The 2 linear equations represent 2 lines in a plane.

Possible solutions:

- Graph: Two lines intersect at exactly one point.
  Solution = intersection point \((x, y)\).
  System is called consistent.
  Equations are called independent.

- Graph: Two parallel lines intersecting at no point.
  No solution. Solution set \(\emptyset\).
  System is called inconsistent.
  Equations are called independent.

- Graph: Two lines overlap as the same line.
  Solution = an infinite number of intersection points.
  Write solution with one variable arbitrary.
  System is called consistent.
  Equations are called dependent.

Substitution Method:

- Isolate 1 variable in 1 equation.
- Substitute into 2nd equation.

Elimination Method:

- The coefficients of the eliminated variable in two equations must be additive inverses
- Multiply one or both equations by a number if needed
- Add equations to eliminate one variable

For three variables: \(x, y, \text{ and } z\)

- The 3 linear equations represent planes in a 3D space.
- Write the solution as a point \((x; y; z)\) (called a triple),
or a set of infinitely many points forming a line,
or a set of infinitely many points forming a plane.

Solving a Linear System with 3 Variables: (3 equations)

- Step 1: Use elimination with a pair of equations to eliminate 1 variable.
- Step 2: Use elimination with a second pair of equations to eliminate the same variable.
- Step 3: Use elimination or substitution to solve 2 resulting equations. Solve for each variable.

Solving a Linear System: (3 variables; ONLY 2 equations)

The intersection of 2 distinct planes will be a line.
Solution: A set of infinitely many points lying on a line.

Goal: Make one variable arbitrary. Find an expression for the remaining variables in terms of the arbitrary variable.

- Step 1: Use elimination to eliminate one variable.
  **NOTE:** Do not eliminate your arbitrary variable.
- Step 2: Solve for one of the 2 remaining variables in terms of the arbitrary variable.
- Step 3: Use an original equation and the expression in Step 2 to solve for the eliminated variable.

Applications - To write a system of equations:

- Step 1: Read the problem carefully.
- Step 2: Assign variables.
- Step 3: Write a system of equations that relates the unknowns.
- Step 4: Solve the system of equations.
- Step 5: State the answer to the problem. Does it seem reasonable?
- Step 6: Check that the answer satisfies the original problem.

Online Homework and Quiz Assignments

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Systems of Linear Equations, 5.1

A system of equations is a SET of equations.

A solution satisfies every equation in the system.

Definition: A linear equation in \( n \) unknowns is any equation of the form \( a_1x_1 + a_2x_2 + \ldots + a_nx_n = b \), for real numbers \( a_1, a_2, \ldots, a_n \) (not all of which are 0) and \( b \).

A linear system is a set of one or more equations containing one or more variables with exponent 1.

Possible solutions:

Graph: Two lines intersect at exactly one point

Solution = intersection point \((x, y)\)

The system is called consistent. There is a solution.
The equations are called independent. Two distinct lines.

Graph: Two parallel lines intersecting at no point.

No solution. Solution = \(\emptyset\)

The system is called inconsistent. There is no solution.
The equations are called independent. Two distinct lines.

Graph: Two lines overlap as the same line.

Solution = an infinite number of intersection points

The system is called consistent. There is a solution.
The equations are called dependent. One resulting line.

Substitution Method:

- Isolate 1 variable in 1 equation.
- Substitute into 2nd equation.

Solve each system by substitution

\#7)

\[ 4x + 3y = -13 \quad \text{Equation #1} \]
\(-x + y = 5 \quad \text{Equation \#2}\)

- Isolate 1 variable in 1 equation.
  We solve equation \#2 for \(y\).
  \[y = 5 + x\]

- Substitute into 2nd equation.
  Now replace \(y\) with \(5 + x\) in equation \#1, and solve for \(x\).
  
  \[
  \begin{align*}
  4x + 3y &= -13 \\
  4x + 3(5 + x) &= -13 \\
  4x + 15 + 3x &= -13 \\
  4x + 3x &= -13 - 15 \\
  7x &= -28 \\
  x &= \frac{-28}{7} = -4
  \end{align*}
  \]

  Use \(y = 5 + x\) to solve for the \(y\)-value.
  \[y = 5 - 4\]
  \[y = 1\]

  Solution = \((-4, 1)\)

  Check:
  
  Equation \#1
  \[
  \begin{align*}
  4x + 3y &= -13 \\
  4(-4) + 3(1) &= -13? \\
  -16 + 3 &= -13 \quad \text{True}
  \end{align*}
  \]

  Equation \#2
  \[
  \begin{align*}
  -x + y &= 5 \\
  -(4) + 1 &= 5? \\
  4 + 1 &= 5 \quad \text{True}
  \end{align*}
  \]

\[\#11\]

\[\begin{align*}
8x - 10y &= -22 & \text{Equation \#1} \\
3x + y &= 6 & \text{Equation \#2}
\end{align*}\]

- Isolate \(y\) in equation \#2.
  \[y = 6 - 3x\]

- Substitute into equation \#1.
  \[
  \begin{align*}
  8x - 10y &= -22 \\
  8x - 10(6 - 3x) &= -22 \\
  8x - 60 + 30x &= -22 \\
  8x + 30x &= -22 + 60 \\
  38x &= 38 \\
  x &= \frac{38}{38} \\
  x &= 1
  \end{align*}
  \]

  Solve for \(y\).
  \[
  \begin{align*}
  y &= 6 - 3(1) \\
  &= 6 - 3 \\
  &= 3
  \end{align*}
  \]

  Solution is \([(1, 3)]\)

  Check:
  
  Equation \#1
  \[
  \begin{align*}
  8x - 10y &= -22 \\
  8(1) - 10(3) &= -22? \\
  8 - 30 &= -22 \quad \text{True}
  \end{align*}
  \]

  Equation \#2
  \[
  \begin{align*}
  3x + y &= 6 \\
  3(1) + 3 &= 6 \quad \text{True}
  \end{align*}
  \]

\[\#15\]

\[\begin{align*}
-2x &= 6y + 18 & \text{Equation \#1} \\
-29 &= 5y - 3x & \text{Equation \#2}
\end{align*}\]
Solving #1 for x yields:
\[ x = \frac{6y + 18}{-2} \]
\[ x = \frac{6y - 2}{-2} \]
\[ x = -3y - 9 \]

Substitute for x in equation #2.

Solve for y:

\[ -29 = 5y - 3x \]
\[ -29 = 5y - 3(-3y - 9) \]
\[ -29 = 5y + 9y + 27 \]
\[ -29 - 27 = 5y + 9y \]
\[ -56 = 14y \]
\[ x = \frac{-11}{1} \]
\[ y = -4 \]

Substitute for y = -4, solve for x:

\[ x = -3y - 9 \]
\[ x = -3(-4) - 9 \]
\[ x = 12 - 9 \]
\[ x = 3 \]

Solution is \{(3, -4)\}

---

**Elimination Method:**

Uses multiplication and addition to eliminate a variable

Allows us to solve for the remaining variable (linear system)

To eliminate a variable,

- the coefficients of that variable in two equations must be additive inverses (same number opposite sign)
- multiply one or both equations by a number
- add or subtract equations to eliminate one variable

**Transformations of a Linear System:**

Operations that may be used to solve the system

- Interchange any two equations of the system.
- Multiply or divide any equation of the system by a nonzero real number.
- Replace any equation of the system by the sum of that equation and a multiple of another equation in the system.

In future courses you'll find that you can organize a system of equations into what is called a matrix. Each row of the matrix consists of the coefficients for one equation.

The first transformation is more meaningful when applied to matrices. We will focus our attention on the second two transformations.

**Solve each system by elimination.**

19)

3x - y = -4 \hspace{1cm} \text{Equation #1}

x + 3y = 12 \hspace{1cm} \text{Equation #2}

To eliminate y, multiply both sides of equation #1 by 3 (the second transformation)

\[ 9x - 3y = -12 \]
\[ x + 3y = 12 \]

Add these equations together (the third transformation)

\[ 9x - 3y = -12 \]
\[ + \quad x + 3y = 12 \]
\[ \hline \]
\[ 10x = 0 \]
\[ x = 0 \]
Technically, the new system is
\[ x = 0 \]
\[ x + 3y = 12 \]

By applying the transformations equation \#1 is replaced by an equivalent equation.
To finish, substitute 0 for \( x \) in either original equation.
\[ 0 + 3y = 12 \]
\[ y = 4 \]

Solution set is \((0, 4)\).

Check:
Equation \#1:
\[ 3x - y = -4 \]
\[ 3(0) - 4 = -4 \text { True} \]

Equation \#2:
\[ x + 3y = 12 \]
\[ 0 + 3(4) = 12 \text { True} \]

25)

\[ 6x + 7y + 2 = 0 \] Equation \#1
\[ 7x - 6y - 26 = 0 \] Equation \#2

To apply the elimination method correctly, organizing the equations into the same format is helpful.

To eliminate \( x \):
Multiply equation \#1 by \(-7\).
Multiply equation \#2 by 6.
\[-42x - 49y - 14 = 0 \]
\[42x - 36y - 156 = 0 \]

Add the 2 equations together
\[-42x - 49y - 14 = 0 \]
\[+ 42x - 36y - 156 = 0 \]
\[0x - 85y - 170 = 0 \]

Solve the resulting equation.
\[-85y - 170 = 0 \]
\[-85y = 170 \]
\[y = \frac{170}{-85} \]
\[y = -2 \]

To finish, substitute \(-2\) for \( y \) in either original equation.
\[6x + 7(-2) + 2 = 0 \]
\[6x - 14 + 2 = 0 \]
\[6x = 0 \]
\[x = 2 \]

Solution set is \((2, -2)\)

39)

\[ \frac{2x}{5} + \frac{12-x}{10} = 1 \] Equation \#1
\[ \frac{2x}{5} + \frac{12-3x}{10} = -3 \] Equation \#2

The equations look complicated because we have the fractions. How might we remove the fractions in one step?

For each equation, remove the fractions by multiplying by the least common denominator.

For equation \#1, LCD = 10. Multiply by 10.
For equation \#2, LCD = 20. Multiply by 20.

Equation \#1:
\[ 10\left(\frac{2x}{5}\right) + 10\left(\frac{12-x}{10}\right) = 10(1) \]
\[2(x + 6) + (2y - x) = 10 \]
\[2x + 12 + 2y - x = 10 \]
\[x + 2y + 12 = 10 \]
\[x + 2y = -2 \]

Equation \#2:
\[ 20\left(\frac{2x^2}{5}\right) + 20\left(\frac{2x+2}{10}\right) = 20(-3) \]
\[5(x + 2) + 4(3y + 2) = -60\]
\[5x + 10 + 12y + 8 = -60\]
\[5x + 12y + 18 = -60\]
\[5x + 12y = -78\]

New system:
\[x + 2y = -2\]
\[5x + 12y = -78\]

To eliminate \(x\): Multiply both sides of equation \#1 by \(-5\).
\[-5x - 10y = 10\]
\[5x + 12y = -78\]

Add the 2 equations together:
\[-5x - 10y + 5x + 12y = 10 + (-78)\]
\[2y = -68\]
\[y = -34\]

Substitute for \(y\) into either original equation.
\[x + 2(-34) = -2\]
\[x - 68 = -2\]
\[x = 66\]

Solution set is \((66, -34)\)

**Solve each system.**

State whether it is inconsistent or has infinitely many solutions. If the system has infinitely many solutions, write the solution set with \(y\) arbitrary.

**31)**

\[9x - 5y = 1 \quad \text{Equation \#1}\]
\[-16x + 10y = 1 \quad \text{Equation \#2}\]

To eliminate \(x\): Multiply both sides of equation \#1 by \(2\).
\[-18x + 10y = 1\]
\[16x - 10y = 2\]

Add the 2 equations together:
\[-18x + 10y + 16x - 10y = 1 + 2\]
\[0x + 0y = 3\]

The resulting equation: \(0 = 3\) FALSE statement!

Therefore, NO SOLUTION. Solution = \(\emptyset\)

The two lines do not intersect. Solve each equation for \(y\):\n\[y = \frac{9}{5}x - \frac{1}{5}\] and \[y = \frac{1}{2}x + \frac{1}{10}\]. Parallel lines.

The system is inconsistent.

**Solve for each system.**

State whether it is inconsistent or has infinitely many solutions. If the system has infinitely many solutions, write the solution set with \(y\) arbitrary.

**33)**

\[4x - y = 0 \quad \text{Equation \#1}\]
\[-8x + 2y = -18 \quad \text{Equation \#2}\]

To eliminate \(x\):
Multiply both sides of equation \#1 by \(2\).
\[-8x + 2y = -18\]
\[8x - 2y = 18\]

Add the 2 equations together:
\[-8x + 2y + 8x - 2y = -18 + 18\]
\[0x + 0y = 0\]
The resulting equation: $0 = 0$ TRUE statement!

Notice:
- Equation #2 is a multiple of equation #1.
  - The graph for both equations is the same line.
  - There are infinitely many intersection points.
Thus there are infinitely many solutions.

NOTE: The solution is NOT all real numbers.
The solution consists of all of the points on the line $4x - y = 9$
The solution set can be expressed in terms of $x$ or $y$.

Directions: Write the solution set with $y$ arbitrary.

Solve equation #1 for $x$.

\[
4x - y = 9 \\
4x = 9 + y \\
x = \frac{9 + y}{4}
\]

Solution set \( \left\{ \left( \frac{9 + y}{4}, y \right) \right\} \)

This solution describes a set of points where the $y$-value can be any real number (arbitrary) and the $x$-value must be $\frac{9 + y}{4}$ so that the point $(x, y)$ lies on the line $4x - y = 9$.

Solving a linear system with 3 variables:

For two variables: $x$ and $y$
- The 2 linear equations represent lines in a 2 dimensional space (2D), the $xy$-plane.
- Write the solution as a point $(x, y)$ or a set of infinitely many points on a line (with one variable arbitrary).

For three variables: $x$, $y$, and $z$
- The 3 linear equations represent planes in a 3 space. (Recall a plane is like an infinitely thin piece of paper that extends to infinity in all directions.)
- Write the solution as a point $(x, y, z)$ called a triple, or a set of infinitely many points forming a line, (with 1 variable arbitrary), or a set of infinitely many points forming a plane. (with 2 variables arbitrary). (Quickly show the diagrams in the book. Highlight the many ways that 3 planes can interact.)
  - (This section is not included in our custom textbook, but may be accessed online via course compass.)

In order to solve a system of equations with $n$ variables and have a numerical value for each variable, we must have $n$ equations. If we have less then $n$ equations, then there must be at least one variable that is left arbitrary.

Solving a Linear System with 3 Variables:
- Step 1: Use elimination with a pair of equations to eliminate 1 variable.
- Step 2: Use elimination with a second pair of equations to eliminate the same variable
- Step 3: Use elimination or substitution to solve 2 resulting equations. Solve for each variable.

Solve each system.

33)

\[
\begin{align*}
2x - 3y - 2z &= -3 & \text{Equation #1} \\
3x + 2y - z &= 12 & \text{Equation #2} \\
-x - y + 4z &= 3 & \text{Equation #3}
\end{align*}
\]

Step 1:
To eliminate $x$: Use elimination with #1 and #3.

Notice: The $x$ coefficients are already additive inverses:

\[
\begin{align*}
x - 3y - 2z &= -3 & \text{Equation #1} \\
-x - 2y + 4z &= 3 & \text{Equation #3} \\
0x - 4y + 2z &= 0
\end{align*}
\]

Resulting equation: $-4y + 2z = 0$. Call this equation #4.

Step 2:
To eliminate $x$ again: Use elimination with #2 and #3.

Multiply both sides of equation #3 by 3.

Add the equations together.
\[ 3x + 2y - z = 12 \quad \text{Equation 2} \]
\[ -3x - 3y + 12z = 9 \quad \text{Equation 3 (multiplied by 3)} \]
\[ -y + 11z = 21 \]

Resulting equation: \(-y + 11z = 21\) Call this equation 5.

**Step 3:**

Use elimination with the 2 resulting equations (#4 and #5).

\[ -4y + 2z = 0 \quad \text{Equation 4} \]
\[ -y + 11z = 21 \quad \text{Equation 5} \]

To eliminate \(y\): Multiply equation #5 by \(-4\).

Add the equations together:

\[ \begin{align*}
-4y + 2z &= 0 \\
+ 4y - 44z &= -84 \\
\hline
-42z &= -84
\end{align*} \]

Solve the resulting equation for \(z\).

\[-42z = -84 \]
\[z = 2\]

To solve for \(y\): Substitute for \(z\) in either #4 or #5.

Substitute for \(z\) in equation #5.

\[ -y + 11z = 21 \]
\[ -y + 11(2) = 21 \]
\[ -y + 22 = 21 \]
\[ -y = -1 \]
\[ y = 1 \]

To solve for \(x\): Substitute for \(z\) and \(y\) in equation #1, #2, or #3.

Substituting for \(z\) and \(y\) in equation #3.

\[ -x - y + 4z = 3 \]
\[ -x - 1 + 4(2) = 3 \]
\[ -x + 8 = 3 \]
\[ -x = -5 \]
\[ x = 4 \]

Solution = \((4, 1, 2)\)

**Solve each system**

**47)**

\[ x + y + z = 2 \quad \text{Equation 1} \]
\[ 2x + y - z = 5 \quad \text{Equation 2} \]
\[ x - y + z = -2 \quad \text{Equation 3} \]

To eliminate \(y\): Use elimination with #1 and #3.

Notice: The \(y\) coefficients are already additive inverses.

\[ \begin{align*}
x + y + z &= 2 \\
+ \quad x - y + z &= -2 \\
\hline
2x + 0y + 2z &= 0
\end{align*} \]

Resulting equation: \(2x + 2z = 0\) Call this equation #4.

To eliminate \(y\) again: Use elimination with #2 and #3.

Notice: The \(y\) coefficients are already additive inverses.

\[ \begin{align*}
2x + y - z &= 5 \\
+ \quad x - y + z &= -2 \\
\hline
3x + 0y + 0z &= 3
\end{align*} \]
Resulting equation: $3x = 3$

$$x = 1$$

To solve for $z$: Substitute for $x$ in equation #4.

$$2x + 2z = 0$$
$$2(1) + 2z = 0$$
$$2z = -2$$
$$z = -1$$

To solve for $y$: Substitute for $x$ and $z$ in any original equation (we'll use equation #1 not for any good reason).

$$x + y + z = 2$$
$$1 + y - 1 = 2$$
$$y = 2$$

The solution set is $\{(1, 2, -1)\}$.

**Solving a Linear System:**

3 variables ONLY 2 equations:

- The intersection of 2 planes will be a line.
- Solution: a set of infinitely many points lying on a line.

Goal: Make one variable arbitrary. Find an expression for the remaining variables in terms of the arbitrary variable. (For example: Let $x$ be the arbitrary variable. Then $x$ is in the solution. Our goal is to solve for $y$ and $z$ in terms of $x$.)

- **Step 1:** Use elimination to eliminate one variable. *NOTE:* Do not eliminate your arbitrary variable.
- **Step 2:** Solve for one of the 2 remaining variables in terms of the arbitrary variable.
- **Step 3:** Use an original equation and the expression in Step 2 to solve for the eliminated variable.

**Solve each system in terms of the arbitrary variable $x$.**

59)

$$x - 2y + 3z = 6 \quad \text{Equation } \#1$$
$$2x + y + 2z = 5 \quad \text{Equation } \#2$$

Let $x$ be the arbitrary variable.

Do not eliminate $x$. Eliminate $y$ or $z$, your choice.

To eliminate $y$: Multiply equation #2 by $-2$.

Add equations together:

$$x - 2y + 3z = 6$$
$$-4x + 2y - 4z = -10$$
$$-3x + 0y - z = -4$$

Resulting equation: $-3x - z = -4$

Solve the resulting equation for $z$ in terms of $x$.

$$-3x - z = -4$$
$$z = -4 + 3x$$

At this point: Solution $= (x, y, 4 - 3x)$

Solution should contain only the arbitrary variable.

Next step: Solve for $y$ in terms of $x$.

To solve for $y$ in terms of $x$: Use substitution.

Substitute for $z$. Recall $z = 4 - 3x$.

Use an original equation.

Using equation #2: $2x - y + 2z = 5$

$$2x - y + 2(4 - 3x) = 5$$
$$2x - y + 8 - 6x = 5$$
\[-4x - y + 8 = 5 \\
-2y = 5 + 4x - 8 \\
-2y = -3 + 4x \\
y = 3 - 4x\]

Solution \((3, 4, 3 - 4a)\)

This triple represents all of the points on the 3-dimensional line. We can substitute any \(x\)-value in this triple and find a point on the solution line where the 2 planes intersect.

**Solve each system.**

State whether it is inconsistent or has infinitely many solutions. If the system has infinitely many solutions, write the solution set with \(z\) arbitrary.

**65)\)

\[
\begin{align*}
3x + 5y - z &= -2 \quad \text{Equation} \#1 \\
4x - y + 2z &= 1 \quad \text{Equation} \#2 \\
-6x - 10y + 2z &= 0 \quad \text{Equation} \#3
\end{align*}
\]

To eliminate \(x\): Use elimination with equation \#1 and \#3.

Multiply equation \#1 by 2.

Add these equations together:

\[
\begin{align*}
6x + 10y - 2z &= -4 \\
-6x - 10y + 2z &= 0 \\
0x + 0y + 0z &= -4
\end{align*}
\]

Resulting equation: \(0 = -4\) **FALSE**

- This implies that there is no intersection between the plane represented by equation \#1 and the plane represented by equation \#3.
- Recall, a solution is the intersection of all 3 planes.
- If \(2\) of the \(3\) planes do not intersect, then there is no intersection between all \(3\) planes. Hence there is no solution to this system.
- Point out the appropriate possible pictures of this system in the textbook.

The system is inconsistent.

**Solve each system.**

State whether it is inconsistent or has infinitely many solutions. If the system has infinitely many solutions, write the solution set with \(z\) arbitrary.

**68)\)

\[
\begin{align*}
2x + y - 3z &= 0 \quad \text{Equation} \#1 \\
4x + 2y - 6z &= 0 \quad \text{Equation} \#2 \\
x - y + z &= 0 \quad \text{Equation} \#3
\end{align*}
\]

To eliminate \(y\): Add equation \#1 and equation \#3.

Notice the coefficients are additive inverses:

\[
\begin{align*}
2x + y - 3z &= 0 \\
+\quad x - y + z &= 0 \\
\underline{3x} &= 0y - 2z = 0
\end{align*}
\]

Resulting equation: \(3x - 2z = 0\) **Equation** \#4

To eliminate \(y\) again: Use elimination with equation \#2 and \#3.

Multiply equation \#3 by 2.

Add the equations together:

\[
\begin{align*}
4x + 2y - 6z &= 0 \\
+\quad 2x - 2y + 2z &= 0 \\
\underline{6x} &= 0y - 4z = 0
\end{align*}
\]

Resulting equation: \(6x - 4z = 0\) **Equation** \#5

Use elimination with the 2 resulting equations (\#4 and \#5).

\[
\begin{align*}
3x - 2z &= 0 \quad \text{Equation} \#4 \\
6x - 4z &= 0 \quad \text{Equation} \#5
\end{align*}
\]

To eliminate \(x\): Multiply equation \#4 by \(-2\)

Add the equations together
\[-6x + 4z = 0\]
\[+ 6x - 4z = 0\]
\[0x + 0z = 0\]

Resulting equation: \(0 = 0\) \textit{TRUE}

A true statement implies infinitely many solutions.

Look back at the original problem. Notice that equation \#2 is a multiple of equation \#1. These 2 equations describe the same (overlapping) plane. These 2 overlapping planes intersect at infinitely many points.

Recall, a solution involves the intersection of all 3 planes. The plane representing equation \#3 must intersect the other(s) at a line.

Write the solution set with \(z\) arbitrary.

Solve for both \(x\) and \(y\) in terms of \(z\).

To solve for \(x\): Use either equation \#4 or equation \#5.

Using equation \#4:

\[3x - 2z = 0\]
\[3x = 2z\]
\[x = \frac{2}{3}z\]

At this point: Solution \(= \left(\frac{2}{3}z, y, z\right)\)

Solution should contain only the arbitrary variable.

Next step: Solve for \(y\) in terms of \(z\).

To solve for \(y\) in terms of \(z\): Use substitution.

Substitute for \(x\). Recall \(x = \frac{2}{3}z\).

Use an original equation.

Using equation \#3:

\[\frac{2}{3}y - y + z = 0\]
\[\frac{2}{3}y - y = 0\]
\[y = \frac{3}{2}z\]

Solution \(= \left(\frac{2}{3}z, \frac{3}{2}z, z\right)\)

This triple represents all of the points on the 3-dimensional line. We can substitute any \(z\) value in this triple and find a point on the solution line where the 3 planes intersect.

\section*{Applications}

\textbf{To write a system of equations:}

- \textbf{Step 1:} Read the problem carefully. Identify what is given and what is to be found.
- \textbf{Step 2:} Assign variables. Write down what each variable represents. Use diagrams or tables as needed.
- \textbf{Step 3:} Write a system of equations that relates the unknowns.
- \textbf{Step 4:} Solve the system of equations.
- \textbf{Step 5:} State the answer to the problem. Does it seem reasonable?
- \textbf{Step 6:} Check the answer in the words of the original problem.

Use a system of equations to solve each problem.

\section*{Example 1}

Find the equation of the parabola \(y = ax^2 + bx + c\) that passes through the points \((2, 3), (-1, 0), (-2, 2)\).

- \textbf{Step 1:} Given \(x\) and \(y\).
  
  Find \(a, b,\) and \(c\).

- \textbf{Step 2:} Let \(a, b,\) and \(c\) represent the coefficients of the quadratic equation. \(y = ax^2 + bx + c\)

- \textbf{Step 3:} Substitute each of the 3 points given.

Each substitution will yield an equation in \(a, b,\) and \(c\):

Substituting the coordinates \((2, 3)\) for \(x\) and \(y\): \(3 = a(2)^2 + b(2) + c\)

\[4a + 2b + c = 3 \quad \text{Equation \#1}\]
Substituting the coordinates \((-1, 0)\) for \(x\) and \(y\) \(0 = a(-1)^2 + b(-1) + c\)

\[ a - b + c = 0 \quad \text{Equation \#2} \]

Substituting the co-ordinates \((-2, 2)\) for \(x\) and \(y\) \(2 = a(-2)^2 + b(-2) + c\)

\[ 4a - 2b + c = 2 \quad \text{Equation \#3} \]

System of Equations:

\[ 4a + 2b + c = 3 \]
\[ a - b + c = 0 \]
\[ 4a - 2b + c = 2 \]

To eliminate \(c\): Use elimination with equation \#1 and \#2.

Multiply equation \#1 by \(-1\).

Add equations together:

\[ -4a - 2b - c = -3 \]
\[ + \quad a - b + c = 0 \]
\[ 
\[ -3a - 3b + 0c = -3 \]

Resulting equation: \(-3a - 3b = -3\) \quad \text{Equation \#4}

To eliminate \(c\): Use elimination with equation \#1 and \#3.

Multiply equation \#1 by \(-1\).

Add equations together:

\[ -4a - 2b - c = -3 \]
\[ + \quad 4a - 2b + c = 2 \]
\[ 0a - 4b + 0c = -1 \]

Resulting equation: \(-4b = -1\)

\[ b = \frac{1}{4} \]

To solve for \(a\): Substitute for \(b\) in equation \#4.

\[ -3a - 3b = -3 \]
\[ -3a - 3\left(\frac{1}{4}\right) = -3 \]
\[ -3a - 3\left(\frac{1}{4}\right) = -3 \]
\[ -3a = -3 + \frac{3}{4} \]
\[ -3a = -\frac{9}{4} \]
\[ a = \frac{3}{4} \]

To solve for \(c\): Substitute for \(a\) and \(b\) in an original equation.

Using equation \#2:

\[ a - b + c = 0 \]
\[ \frac{3}{4} - \frac{1}{4} + c = 0 \]
\[ \frac{1}{2} + c = 0 \]
\[ c = -\frac{1}{2} \]

Thus \(y = ax^2 + bx + c\) becomes \(y = \frac{3}{4}x^2 + \frac{1}{2}x - \frac{1}{2}\)

90) Costs of Goats and Sheep

At the Jeb Bush ranch, 6 goats and 5 sheep sell for $305, while 2 goats and 9 sheep sell for $285. Find the cost of a single goat and of a single sheep.

Let

\[ x = \text{cost of a single goat} \]
\[ y = \text{cost of a single sheep} \]

6 goats and 5 sheep sell for $305 translates to:

\[ 6x + 5y = 305 \]

2 goats and 9 sheep selling for $285 translates to:

\[ 2x + 9y = 285 \]
System of equations:

\[ \begin{align*}
6x + 5y &= 305 \\
2x + 9y &= 285
\end{align*} \]  
\text{Equation } #1\text{ and } #2

Use either the elimination method or substitution to solve for \( x \) and \( y \):

To eliminate \( x \): Multiply equation \#2 by \(-3\).

Add the equations together:

\[
\begin{align*}
6x + 5y &= 305 \\
-6x - 27y &= -855 \\
0x - 22y &= -550
\end{align*}
\]

Solve the resulting equation for \( y \): \( 0x - 22y = -550 \)

\[ -22y = -550 \]

\[ y = 25 \]

Substitute for \( y \) in one of the original equations: \( 2x + 9y = 285 \)

\[ 2x + 9(25) = 285 \]

\[ 2x = 60 \]

\[ x = 30 \]

The cost of a single goat is \$30.
The cost of a single sheep is \$25.

98) Investment Decisions.

Jane Hook invests \$40,000 received in an inheritance in three parts. With one part she buys mutual funds that offer a return of 2% per year. The second part, which amounts to twice the first, is used to buy government bonds paying 2.5% per year. She puts the rest of the money into a savings account that pays 1.25% annual interest. During the first year, the total interest is \$825. How much did she invest at each role?

Let

\[ x = \$x \text{ invested in mutual funds} \]
\[ y = \$y \text{ invested in government bonds} \]
\[ z = \$z \text{ invested in a savings account} \]

Use the facts given to write 3 equations:

- Interest earned: \( 0.02x + 0.025y + 0.0125z = 825 \) \(#1\)
- Total invested: \( x + y + z = 40000 \) \(#2\)
- Twice the first: \( y = 2x \) \(#3\)

Substitute for \( y \) in equation \#2, using expression in \#3:

\[ x + 2x + z = 40000 \]

\[ 3x + z = 40000 \] Equation \#4

Substitute for \( y \) in equation \#1, using expression in \#3:

\[ 0.02x + 0.025(2x) + 0.0125z = 825 \]

\[ 0.02x + 0.05x + 0.0125z = 825 \]

\[ 0.07x + 0.0125z = 825 \] Equation \#5

To eliminate \( z \):

- Use elimination with equation \#4 and \#6.
- Multiply equation \#4 by \(-0.0125\).
- Add these equations together

\[
\begin{align*}
-0.0375x - 0.0125z &= -500 \\
+ & 0.07x + 0.0125z = 825 \\
0.0325x &= 325
\end{align*}
\]

Solving for \( x \):

\[ 0.0325x = 325 \]

\[ x = \frac{325}{0.0325} \]

\[ x = 10,000 \]

Solving for \( y \):

\[ y = 2(10,000) \]

\[ y = 20,000 \]

Solving for \( z \):

\[ 10,000 + 20,000 + z = 40,000 \]
$x = 10,000$

Jane invested:
$10,000$ at $2\%$ rate,
$20,000$ at $2.5\%$ rate, and
$10,000$ at $1.25\%$ rate.

95) Triangle Dimensions

The perimeter of a triangle is $59$ in. The longest side is $11$ in. longer than the medium side, and the medium side is $3$ in. more than the shortest side. Find the length of each side of the triangle.

Let $x =$ length of longest side
$y =$ length of medium side
$z =$ length of shortest side

Perimeter: $x + y + z = 59$  Equation #1

Longest side is $11$ in. longer than the medium side:

$x = 11 + y$  Equation #2

Medium side is $3$ in. more than the shortest side:

$y = 3 + z$  Equation #3

System of Equations:

\[
x + y + z = 59
\quad x = 11 + y
\quad y = 3 + z
\]

Solve by substitution:

Rewrite equation #3 and solve for $z$.

\[
z = y - 3
\]

Now substitute for $x$ and $z$ in equation #1 and solve for $y$.

\[
x + y + z = 59
\quad (11 + y) + y + (y - 3) = 59
\]

\[
8 + 3y = 59
\quad 3y = 51
\quad y = 17
\]

To solve for $x$: Substitute into equation #2.

\[
x = 11 + y x = 11 + 17 x = 18
\]

To solve for $z$: Substitute into equation #3.

\[
z = y - 3 z = 17 - 3 z = 14
\]

Solution: $(28, 17, 14)$

Check:

Does adding all the sides give us the perimeter of $59$ inches?

$28 + 17 + 14 = 59$ TRUE
Week 15

This week is reserved for review of the course material, information for the final exam expectations, and grade calculations.

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Online Homework and Quiz Assignments

Log into MyLabPlus and finish working on your cumulative homework and quiz due this week. Then go to www.ucf.mylabsplus.com and begin working on your assignments.
Week 16

You will be taking your final exam in the MALL this week. Good luck on your final exams!